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STABILIZATION OF LINEAR UNCERTAIN DELAY SYSTEMS WITH ANTISYMMETRIC STEPWISE CONFIGURATIONS

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Abstract. This paper investigates the problem of designing a linear memoryless state feedback control to stabilize a class of linear uncertain systems with state delays. Each uncertain parameter and each delay under consideration might vary with time in an arbitrarily large range. In such a situation, the locations of uncertain elements in the system matrices play an important role. Wei introduced the concept of antisymmetric stepwise configuration (ASC) and proved that it is a necessary and sufficient condition for linear uncertain systems to be quadratically stabilizable using linear state feedback control to have this configuration. However, his method is inapplicable to systems that contain delays in the state variables. On the other hand, Amemiya developed conditions for the stabilization of linear uncertain systems with state delays using linear memoryless state feedback control. This paper presents development of the conditions of this problem that have been obtained to date. Fundamentally, it is proved that having an ASC is also a sufficient condition for the stabilization of linear uncertain delay systems. For systems satisfying the stabilizability conditions, a simple control design procedure is also provided and illustrated by an example.

1. Introduction

Generally, it is difficult to describe the dynamic behaviors of real systems precisely using mathematical equations that consist only of known constant parameters since systems often have an inherent uncertainty. For governing many dynamical systems, a stabilization method of mathematical system models that include uncertain parameters must be developed. For this reason, a considerable attention has been paid to research into robust stabilization of uncertain systems [3, 4, 10, 13, 19]. In most instances, physical, chemical, biological, and economical phenomena depend naturally not only on the present state but also on past occurrences. The importance of time delays in the stability analysis is well recognized in a wide range of applications [7, 9, 17, 20]. Therefore, this paper specifically examines the stabilization problem of uncertain delay systems.

In particular, the stabilization problem of linear uncertain delay systems using linear memoryless state feedback control is emphasized since it can be regarded as basic and realistic. Moreover, the systems under consideration contain time-varying uncertain entries in the system matrices and time-varying uncertain delays in the state variables. Each value of uncertain entries and delays may vary independently in an arbitrarily large range. In this situation, the locations of uncertain elements in the system matrices play an important role. This paper presents investigation of the permissible locations of uncertain elements, which are allowed to take unlimited large values, for the stabilization using linear state feedback control.

It is useful to classify the existing results on the stabilization of uncertain systems into two categories. The first category includes several results which provide the stabilizability conditions depending on the ranges of uncertain parameters. The results in the second category provide the stabilizability conditions that are independent of the ranges of uncertain elements but which depend on their locations. This paper specifically addresses the second category.

For uncertain systems with delays, the Lyapunov stability approach with the Krasovskii-based or Razumikhin-based method is a commonly used tool. The stabilization problem has been reduced to solving linear matrix inequalities (LMI) [6,14,16], which necessitates complicated numerical calculations on the algorithm, even though a convenient software package is available. Moreover, LMI conditions fall into the first category; for this reason, they are often used to determine the permissible ranges of uncertain parameters for the stabilization. When the ranges of uncertain parameter values exceed a certain value, LMI solver becomes infeasible. In such cases, guidelines for redesigning the controller are usually lacking.

On the other hand, the stabilizability conditions provided here can be easily verified merely by examining the uncertainty locations in given system matrices. Once a system satisfies the stabilizability conditions, a stabilizing controller can be constructed, irrespective of the number of uncertain parameters. The control design procedure provided here is a simple algorithm that includes no complex calculations such as convex optimization. We can redesign the controller for improving robustness merely by modifying the design parameter when the uncertain parameters exceed the upper bounds given beforehand.

In the second category, the stabilization problem of linear uncertain systems without delays was solved by Wei [21, 22]. The stabilizability conditions have a particular geometric configuration with respect to the permissible locations of uncertain elements. For the quadratic stabilization, Wei [21] introduced the concept of antisymmetric stepwise configuration (ASC) and proved that, to have this configuration, it is necessary and sufficient for linear time-varying uncertain systems to be quadratically stabilizable using linear state feedback control. In [21], however, the conditions with ASC are constructed based on a quadratic Lyapunov function; they are not applicable to systems that contain delays in the state variables.

On the contrary, based on the properties of an M-matrix, Amemiya [2] developed conditions for the stabilization of linear time-varying uncertain systems with time-varying delays using linear memoryless state feedback control. To obtain these conditions, Amemiya used the convergence conditions of solutions of delay differential inequalities, which are free from the restriction of Lyapunov functions. The conditions obtained in [2] show a similar configuration to an ASC, but the allowable uncertainty locations are fewer than in an ASC by one step. It has also been reported [1] that such a configuration can be developed to a basic form of ASC. However, in [1] the uncertainties are restricted to the system parameters ΔA without including the input coefficient Δb, whereas the original ASC consists of both ΔA and Δb.

In this paper, the conditions from [1, 2] are developed. It is proved that having an ASC is also a sufficient condition for the stabilization of linear uncertain delay systems using linear state feedback control. In other words, the previous conditions are developed so that the allowable uncertainty locations for the stabilization of linear uncertain systems with delays can be increased to that for the quadratic stabilization of these systems without delays. It is shown that if a system satisfies such developed conditions, then the system is stabilizable via a linear control, irrespective to the upper bounds of both uncertain parameters and delays. To achieve our objective in this paper, the means of constructing the Vandermonde matrix and choosing the eigenvalues are revised considerably as compared to those of the previous works [1, 2], which is meaningful progress.

This paper is organized as follows. Some notation and terminology are given in Sec. 2. The system considered here is defined in Sec. 3. Section 4 is devoted to the introduction of a stability criterion for a delay differential equation and to obtaining the preliminary results for the present problem. The main result is provided in Sec. 5. A brief algorithm for the construction of the control is given in Sec. 6 to aid understanding of the proposed method. An illustrative example is given in Sec. 7. Finally, some concluding remarks are presented in Sec. 8.

2. Notation and Terminology

First, some notation and terminology used in the subsequent description are given. For $a, b \in \mathbb{R}^m$ or $A, B \in \mathbb{R}^{n\times m}$, every inequality between a and b or A and B such as $a > b$ or $A > B$ indicates that it is satisfied componentwise by a and b or A and B. If $A \in \mathbb{R}^{n\times m}$ satisfies $A \geq 0$, A is called a non-negative matrix. The transpose and the determinant of $A \in \mathbb{R}^{n\times m}$ are denoted by A' and $\det(A)$, respectively. For $a = (a_1, \ldots, a_m)' \in \mathbb{R}^m$, $|a| \in \mathbb{R}^m$ is defined as $|a| = (|a_1|, \ldots, |a_m|)'$. Also for $A = (a_{ij}) \in \mathbb{R}^{n\times m}$, $|A|$ denotes a matrix with $|a_{ij}|$ as its (i, j) entries. For $A = (a_{ij}) \in \mathbb{R}^{n\times m}$, the matrix $B = (b_{ij}) \in \mathbb{R}^{n\times m}$, which is defined as

$$b_{ij} = \begin{cases} |a_{ij}|, & i \neq j, \\ a_{ij}, & i = j, \end{cases}$$

is denoted by $|A|_q$. Let $\mathrm{diag}\{\cdots\}$ denote a diagonal matrix. Let $[a, b]$, a,$b \in \mathbb{R}$ be an interval in $\mathbb{R}$. The set of all continuous or piecewise continuous functions with the domain $[a, b]$ and range $\mathbb{R}^n$ is denoted, respectively, by $\mathcal{C}^n[a, b]$ or $\mathcal{D}^n[a, b]$. We denote it simply by $\mathcal{C}^n$ or $\mathcal{D}^n$ if the domain is $\mathbb{R}$.

Definition 1. A real square matrix all of whose off-diagonal entries are nonpositive is called an M-matrix if it is nonsingular and its inverse matrix is nonnegative. The set of all M-matrices is denoted by $\mathcal{M}$.

The notation for a class of functions is introduced below. Let $\xi(\mu) \in \mathcal{C}^1$ and let $m \in \mathbb{R}$ be a constant. If $\xi(\mu)$ satisfies the conditions

$$\begin{aligned} \left|\frac{\xi(\mu)}{\mu^m}\right| < \infty \quad \text{as} \quad |\mu| \to \infty, \\ \left|\frac{\xi(\mu)}{\mu^{m-a}}\right| \to \infty \quad \text{as} \quad |\mu| \to \infty \end{aligned} \tag{2.1}$$

for any positive scalar $a \in \mathbb{R}$, then $\xi(\mu)$ is called a function of order m, and we denote this as follows:

$$\mathrm{Ord}(\xi(\mu)) = m.$$

The set of all $\mathcal{C}^1$ functions of order m is denoted by $O(m)$,

$$O(m) = \left\{\xi(\mu) \mid \xi(\mu) \in \mathcal{C}^1,\ \mathrm{Ord}(\xi(\mu)) = m\right\}. \tag{2.2}$$

Also, it is worth to note that m can be a negative number and that the following relations between $\xi_1(\mu) \in O(m_1)$ and $\xi_2(\mu) \in O(m_2)$ hold:

$$\begin{aligned} \mathrm{Ord}(\xi_1(\mu) \pm \xi_2(\mu)) &= \max\{m_1, m_2\}, \\ \mathrm{Ord}(\xi_1(\mu) \times \xi_2(\mu)) &= m_1 + m_2, \\ \mathrm{Ord}(\xi_1(\mu)/\xi_2(\mu)) &= m_1 - m_2. \end{aligned}$$

3. Description of the system

Let n be a fixed positive integer. The system considered here is given by a delay differential equation defined on $x \in \mathbb{R}^n$ for $t \in [t_0, \infty)$ as follows:

$$\dot{x}(t) = A^0 x(t) + \Delta A^1(t) x(t) + \sum_{i=1}^{r} \Delta A^{2i}(t) x(t - \tau_i(t)) + (b + \Delta b(t)) u(t), \quad (3.1)$$

with an initial curve $\phi \in \mathcal{D}^n[t_0 - \tau_0, t_0]$. Here, A^0, $\Delta A^1(t)$, and $\Delta A^{2i}(t)$ $(i = 1, \ldots, r)$ are all real $(n \times n)$-matrices, where r is a fixed positive integer; also, A^0 is a known constant matrix. Furthermore, $\Delta A^1(t)$ and $\Delta A^{2i}(t)$ $(i = 1, \ldots, r)$ are uncertain coefficient matrices and may vary with $t \in [t_0, \infty)$. Other variables are as follows: $u(t) \in \mathbb{R}$ is a control variable, $b \in \mathbb{R}^n$ is a known constant vector, and $\Delta b(t) \in \mathbb{R}^n$ is an uncertain coefficient vector which may vary with $t \in [t_0, \infty)$. In addition, all $\tau_i(t)$ $(i = 1, \ldots, r)$ are piecewise continuous functions and are uniformly bounded, i.e., for a nonnegative constant τ_0 they satisfy

$$0 \le \tau_i(t) \le \tau_0, \quad i = 1, \quad , r, \quad (3.2)$$

for all $t \ge t_0$. The upper bound τ_0 can be arbitrarily large and is not necessarily assumed to be known.

It is assumed that all entries of $\Delta A^1(t)$, $\Delta A^{2i}(t)$, and $\Delta b(t)$ are piecewise continuous functions and are uniformly bounded, i.e., for a nonnegative constant matrices ΔA^{10}, $\Delta A^{2i0} \in \mathbb{R}^{n \times n}$, and for a nonnegative constant vector $\Delta b^0 \in \mathbb{R}^n$, they satisfy

$$\left|\Delta A^1(t)\right| \le \Delta A^{10}, \quad \left|\Delta A^{2i}(t)\right| \le \Delta A^{2i0}, \quad |\Delta b(t)| \le \Delta b^0 \quad (3.3)$$

for all $t \ge t_0$. The upper bound of each entry can independently take an arbitrarily large value, but each is assumed to be known.

Assumption 1. The pair (A^0, b) of the nominal system is a controllable pair and is in the controllable canonical form. Then A^0 and b are given as follows:

$$A^0 = \begin{pmatrix} 0 & 1 & 0 & \ldots & \ldots \\ 0 & 0 & 1 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & 1 \\ 0 & 0 & \ldots & \ldots & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \quad (3.4)$$

Definition 2. System (3.1) is said to be stabilizable via a linear control if there exists a linear memoryless state feedback control $u(t) = g'x(t)$, $g \in \mathbb{R}^n$ such that the equilibrium point $x = 0$ of the resulting closed-loop system is uniformly and asymptotically stable for all admissible uncertain delays (3.2) and uncertain parameters (3.3).

4. Preliminaries

This section provides some preliminary tools that are necessary to derive the main result. First, a stability criterion for a delay differential equation in a little more general setting than system (3.1) is presented in Sec. 4.1. Based on such a stability criterion, we next consider the stabilization problem of system (3.1) using the linear control and linear transformation. A basic condition for the stabilizability of system (3.1) is given in Sec. 4.2. A particular geometric configuration obtained previously for system (3.1) to satisfy such a stabilizability condition is introduced in Sec. 4.3.

4.1. **Stability criterion.** Consider the following delay differential equation defined on $x \in \mathbb{R}^n$ for $t \in [t_0, \infty)$:

$$\dot{x}(t) = Hx(t) + \Delta H^1(t)x(t) + \sum_{i=1}^{r} \Delta H^{2i}(t)x(t - \tau_i(t)) \tag{4.1}$$

with an initial curve $\phi \in \mathcal{D}^n[t_0 - \tau_0, t_0]$. Here H, $\Delta H^1(t)$, $\Delta H^{2i}(t)$ $(i = 1, \dots, r)$ are real $(n \times n)$-matrices, where r is a fixed positive integer; also, $\tau_i(t)$ $(i = 1, \dots, r)$ are assumed to satisfy condition (3.2), and $\Delta H^1(t)$ and $\Delta H^{2i}(t)$ $(i = 1, \dots, r)$ can vary with $t \in [t_0, \infty)$. It is assumed that all of their entries are piecewise continuous functions and are uniformly bounded, i.e., for given constant nonnegative matrices $H^{10}, H^{2i0} \in \mathbb{R}^{n\times n}$ (i=1,...,r), they satisfy the inequalities

$$\left|\Delta H^1(t)\right| \le H^{10}, \quad \left|\Delta H^{2i}(t)\right| \le H^{2i0} \tag{4.2}$$

for all $t \ge t_0$.

The problem related to the existence and uniqueness of solutions of this equation has been solved in [2,8,11,12]. Moreover, the following lemma was proved in [2].

Lemma 1 (see [2]). *Assume that*

$$\left(- |H|_q - H^{10} - \sum_{i=1}^{r} H^{2i0} \right) \in \mathcal{M}. \tag{4.3}$$

Then, every solution of Eq. (4.1) *converges uniformly and exponentially to the equilibrium point* $x = 0$*, independently of delays.*

The stability criterion given in Lemma 1 plays a crucial role in this study. In what follows, we recall the properties of M-matrices to introduce useful propositions for the proof of the main theorem.

Lemma 2 (see [18]). *Let* A, $B \in \mathbb{R}^{n\times n}$ *be constant matrices. Assume that* $A \ge B$. *For any* $C \in \mathbb{R}^{n\times n}$, $(C - B) \in \mathcal{M}$ *if* $(C - A) \in \mathcal{M}$, *provided that all off-diagonal entries of* $(C - B)$ *are nonpositive.*

Lemma 3 (see [5]). *Assume that all off-diagonal entries of $A \in \mathbb{R}^{n\times n}$ are nonpositive. Then A is an M-matrix if and only if all the upper principal minors of A are positive*:

$$\det \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1i} \\ a_{21} & a_{22} & \dots & a_{2i} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ii} \end{pmatrix} > 0, \quad i = 1, 2, \dots, n. \tag{4.4}$$

Let $A_{i-1} \in \mathbb{R}^{(i-1)\times(i-1)}$, $b \in \mathbb{R}^{i-1}$, $c' \in \mathbb{R}^{i-1}$, and $d \in \mathbb{R}$ be given from a matrix $A_i \in \mathbb{R}^{i\times i}$ in the following form:

$$A_i = \left(\begin{array}{c|c} A_{i-1} & b \\ \hline c & d \end{array} \right). \tag{4.5}$$

Let $f : \mathbb{R}^{i\times i} \mapsto \mathbb{R}^{(i-1)\times(i-1)}$ be an operator defined for $i \geq 2, d \neq 0$ as follows:

$$f(A_i) = A_{i-1} - bd^{-1}c. \tag{4.6}$$

For convenience, we introduce the following notation:

$$f^0(A_i) = A_i \in \mathbb{R}^{i\times i}, \tag{4.7}$$

$$f^1(A_i) = f(A_i) \in \mathbb{R}^{(i-1)\times(i-1)}, \tag{4.8}$$

$$f^2(A_i) = f(f(A_i)) \in \mathbb{R}^{(i-2)\times(i-2)}, \quad \dots, \tag{4.9}$$

$$f^{i-1}(A_i) = \underbrace{f(f(\cdots f(A_i)\cdots))}_{i-1} \in \mathbb{R}. \tag{4.10}$$

The following proposition enables verification of whether a given matrix is an M-matrix by iterative calculations.

Proposition 1. *Assume that $A \in \mathbb{R}^{n\times n}$ is a matrix all of whose off-diagonal entries are nonpositive. If all diagonal entries of $f^p(A)$ are positive for all $p = 0, \dots, n-1$, then A is an M-matrix.*

Proof. First, we show that the following statement holds:

$$f^{i+1}(A) \in \mathcal{M} \quad \Rightarrow \quad f^i(A) \in \mathcal{M}. \tag{4.11}$$

Considering $f^i(A)$ as the matrix having the form shown in (4.5), we can rewrite (4.11) as

$$(A_{i-1} - bd^{-1}c) \in \mathcal{M} \quad \Rightarrow \quad A_i \in \mathcal{M}. \tag{4.12}$$

Since all elements of $bd^{-1}c$ are nonnegative, it is obvious from Lemma 2 that

$$(A_{i-1} - bd^{-1}c) \in \mathcal{M} \quad \Rightarrow \quad A_{i-1} \in \mathcal{M}. \tag{4.13}$$

For a matrix A_i in (4.5), the following equivalence holds (see [15]):

$$\det(A_i) = \det(d)\det(A_{i-1} - bd^{-1}c). \tag{4.14}$$

If $(A_{i-1} - bd^{-1}c) \in \mathcal{M}$, then $\det(A_{i-1} - bd^{-1}c) > 0$. Moreover, if all diagonal entries of $f^p(A)$ are positive for all $p = 0, \ldots, n-1$, then $\det(d) > 0$. Under these assumptions, we can find $\det(A_i) > 0$ from (4.14). Using Lemma 3, it turns out from $A_{i-1} \in \mathcal{M}$ that all upper principal minors of A_{i-1} are positive. Hence it is obvious from the inequality $\det(A_i) > 0$ that all upper principal minors of A_i are positive: $A_i \in \mathcal{M}$. Therefore, if $(A_{i-1} - bd^{-1}c) \in \mathcal{M}$, then $A_i \in \mathcal{M}$. Simultaneously, it can be seen that statement (4.11) holds.

Next, under the assumption $f^{n-1}(A) \in \mathbb{R}{>}\,0$, we obtain $f^{n-1}(A) \in \mathcal{M}$. Using relation (4.11), it is obvious that

$$\begin{aligned} f^{n-1}(A) \in \mathcal{M} \quad &\Rightarrow \quad f^{n-2}(A) \in \mathcal{M} \quad \Rightarrow \ldots \\ &\Rightarrow \quad f^{1}(A) \in \mathcal{M} \quad \Rightarrow \quad f^{0}(A) \in \mathcal{M}. \end{aligned} \tag{4.15}$$

Consequently, if all diagonal entries of $f^p(A)$ are positive, $p = 0, \ldots, n-1$, then A is an M-matrix. $\square$

From Proposition 1, we obtain the following auxiliary propositions using the notation in Eq. (2.2).

Proposition 2. *Let k be an integer satisfying $1 < k < n$. Let $A \in \mathbb{R}^{n\times n}$ be a diagonal matrix all of whose entries are positive, and let $B \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. Let A and B be decomposed into four block matrices as follows:*

$$A = \left(\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & A_{22} \end{array}\right), \quad B = \left(\begin{array}{c||c} \boxed{B_{11}} & \boxed{B_{12}} \\ \hline \boxed{B_{21}} & \boxed{B_{22}} \end{array}\right), \tag{4.16}$$

where all entries of each block matrix are functions of μ of the same order. In addition, A_{11} and A_{22} represent, respectively, $k \times k$ and $(n-k)\times(n-k)$ diagonal matrices. All entries of A_{11} and A_{22} belong to $O(a_{11})$ and $O(a_{22})$, respectively. In addition, B_{11}, B_{12}, B_{21}, and B_{22} denote $k \times k$, $k \times (n-k)$, $(n-k)\times k$, and $(n-k)\times(n-k)$ block matrices, respectively. All entries of B_{11}, B_{12}, B_{21}, and B_{22} belong to $O(b_{11})$, $O(b_{12})$, $O(b_{21})$, and $O(b_{22})$, respectively. For sufficiently large μ, if

$$a_{11} > b_{11}, \quad a_{22} > b_{22}, \quad a_{11} > b_{12} - a_{22} + b_{21}, \tag{4.17}$$

then the matrix $C = A - B$ is an M-matrix.

Proof. The proof is obtained by a straightforward computation of $f^p(C)$ $(p = 0, \ldots, n-1)$ in Proposition 1. $\square$

Proposition 3. *Let k_1, k_2 be integers satisfying $1 < k_1 < k_2 < n$. Let $A \in \mathbb{R}^{n\times n}$ be a diagonal matrix all of whose entries are positive, and let $B \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. Let A and B be decomposed into nine block matrices as follows:*

$$A = \left(\begin{array}{c|c|c} A_{11} & 0 & 0 \\ \hline 0 & A_{22} & 0 \\ \hline 0 & 0 & A_{33} \end{array}\right), \quad B = \left(\begin{array}{c|c|c} \boxed{B_{11}} & \boxed{B_{12}} & \boxed{B_{13}} \\ \hline \boxed{B_{21}} & \boxed{B_{22}} & \boxed{B_{23}} \\ \hline \boxed{B_{31}} & \boxed{B_{32}} & \boxed{B_{33}} \end{array}\right). \tag{4.18}$$

Here, all the first, second, and third row blocks represent matrices with k_1, (k_2-k_1), and $(n-k_2)$ rows, respectively, and all the first, second, and third column blocks represent matrices with k_1, (k_2-k_1), and $(n-k_2)$ columns, respectively. Furthermore, all entries of each block matrix are functions of μ of the same order. All entries of the diagonal matrices A_{11}, A_{22}, and A_{33} belong to $O(a_{11})$, $O(a_{22})$, and $O(a_{33})$, respectively. All entries of each block matrix B_{11}, B_{12}, B_{13} B_{21}, B_{22}, B_{23} B_{31}, B_{32}, and B_{33}, belong to $O(b_{11})$, $O(b_{12})$, $O(b_{13})$, $O(b_{21})$, $O(b_{22})$, $O(b_{23})$, $O(b_{31})$, $O(b_{32})$, and $O(b_{33})$, respectively. For sufficiently large μ, if

$$\begin{aligned} &a_{11} > b_{11}, \qquad a_{11} > b_{12} - a_{22} + b_{21}, \\ &a_{22} > b_{22}, \qquad a_{11} > b_{13} - a_{33} + b_{31}, \\ &a_{33} > b_{33}, \qquad a_{22} > b_{23} - a_{33} + b_{32}, \\ &a_{11} > \max\{b_{12}, (b_{13} + b_{32} - a_{33})\} - a_{22} + \max\{b_{21}, (b_{23} + b_{31} - a_{33})\}, \end{aligned} \tag{4.19}$$

then the matrix $C = A - B$ is an M-matrix.

Proof. The proof is obtained by a straightforward computation of $f^p(C)$ $(p = 0, \dots, n-1)$ in Proposition 1. □

For the evaluation of stability condition (4.3) in Lemma 1, Propositions 2 and 3 are useful. They are used in the proof of the main theorem.

4.2. **Construction of control.** Now we consider the stabilization problem of system (3.1) based on the stability criterion from Lemma 1. We investigate whether is it possible to construct a linear memoryless state feedback control,

$$u(t) = g'x(t), \tag{4.20}$$

using a constant vector $g \in \mathbb{R}^n$, which assures the uniform convergence of the solutions of Eq. (3.1). Consider a linear transformation such that

$$\begin{cases} v(t) = T^{-1}x(t), \\ v(t-\tau_i(t)) = T^{-1}x(t-\tau_i(t)), \end{cases} \tag{4.21}$$

using a real nonsingular $(n \times n)$-matrix T. Substituting Eqs. (4.20) and (4.21) into Eq. (3.1), we obtain the following equation:

$$\dot{v}(t) = T^{-1}(A^0 + bg')Tv(t) + T^{-1}\Delta A^1(t)Tv(t) + \sum_{i=1}^{r} T^{-1}\Delta A^{2i}(t)Tv(t - \tau_i(t)) + T^{-1}\Delta b(t)g'Tv(t). \quad (4.22)$$

Because of Assumption 1, it is possible to choose $g \in \mathbb{R}^n$ such that all eigenvalues of $(A^0 + bg')$ are real, negative, and distinct. Let g be chosen in such a way. In addition, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be such eigenvalues of $(A^0 + bg')$. Let T from Eq. (4.21) be the Vandermonde matrix constructed using $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$T = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{pmatrix}. \quad (4.23)$$

This T is well known to be nonsingular in view of the above assumption. Define Λ as follows:

$$\Lambda = T^{-1}(A^0 + bg')T = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}. \quad (4.24)$$

Define H, H^1, H^{2i} $(i = 1, \ldots, r)$ as follows:

$$H = \Lambda, \quad (4.25)$$

$$H^1(t) = T^{-1}\Delta A^1(t)T + T^{-1}\Delta b(t)g'T, \quad (4.26)$$

$$H^{2i}(t) = T^{-1}\Delta A^{2i}(t)T. \quad (4.27)$$

Then, Eq. (4.22) takes the form

$$\dot{v}(t) = Hv(t) + H^1(t)v(t) + \sum_{i=1}^{r} H^{2i}(t)v(t - \tau_i(t)). \quad (4.28)$$

For the coefficients of Eq. (4.28), we obtain the following relations:

$$|H|_q = |\Lambda|_q = \Lambda, \quad (4.29)$$

$$\left|H^1(t)\right| \le \left|T^{-1}\right| \Delta A^{10} |T| + \left|T^{-1}\right| \Delta b^0 |g'| |T|, \quad (4.30)$$

$$\sum_{i=1}^{r} \left|H^{2i}(t)\right| \le \left|T^{-1}\right| \Delta A^{20} |T|, \quad (4.31)$$

where ΔA^{20} in inequality (4.31) is defined by

$$\Delta A^{20} = \sum\nolimits_{i=1}^{r} \Delta A^{2i0}.$$

Now, let P be defined as follows:

$$P = -\Lambda - \left|T^{-1}\right| \Delta A^{30} |T| - \left|T^{-1}\right| \Delta b^0 |g'| |T|. \quad (4.32)$$

Here, $\Delta A^{30} := \Delta A^{10} + \Delta A^{20}$.

From Lemma 1, we obtain the following proposition.

Proposition 4. *If there exists T in Eq.* (4.23) *which assures*

$$P \in \mathcal{M}, \tag{4.33}$$

then system (3.1) *is stabilizable via a linear control.*

Note that our problem has been reduced to finding T that enables P to satisfy condition (4.33). In the subsequent discussion, we consider the possibility of choosing T that assures $P \in \mathcal{M}$.

4.3. **Configuration of uncertain element.** It was shown in [2] that if system (3.1) has a particular geometric configuration with respect to the locations of uncertain elements in ΔA^1, ΔA^{2i} $(i = 1, \ldots, r)$, then there exists some T which assures $P \in \mathcal{M}$. First, we introduce a set of matrices $\Omega(k) \subset \mathbb{R}^{n\times n}$ to prove preliminary results.

Definition 3. Let k be an integer satisfying $0 \le k \le n$. For this k, let $\Omega(k) = \{D = (d_{ij}) \in \mathbb{R}^{n\times n}\}$ be a set of matrices with the following properties:

(i) if $1 \le j \le k+1$, then $d_{ij} = 0$ for $j-1 \le i \le 2k-j+1$;
(ii) if $k+2 \le j \le n$, then $d_{ij} = 0$ for $2k-j+1 \le i \le j-1$.

On the existence of T which assures $P \in \mathcal{M}$, the following lemma has been proved [2].

Lemma 4 (see [2])**.** *If $\Delta A^{30} \subset \Omega(k)$ for fixed k, then it is possible to choose T from* (4.23) *such that $P \in \mathcal{M}$ is satisfied.*

ΔA^{30} is defined in Eq. (4.32). Note that if an entry of ΔA^{30} is zero, then all the corresponding entries of $\Delta A^1(t)$ and $\Delta A^{2i}(t)$ $(i = 1, \ldots, r)$ are equal to zero for all $t \ge t_0$. For this reason, all uncertain coefficient matrices must satisfy the same structure condition simultaneously.

Note that if the locations of uncertain elements in ΔA^{30} satisfy the conditions of $\Omega(k)$, then system (3.1) is stabilizable through a proper choice of g such that T assures $P \in \mathcal{M}$. Actually, $\Omega(k)$ has a configuration similar to an antisymmetric stepwise configuration (ASC) defined in [21]. Compared to an ASC, the permissible locations of uncertain elements of $\Omega(k)$ are fewer than those of an ASC by one step, as is shown in Figs. 1 and 2. Related details will be discussed in the next section.

5. Main results

In this section, first, the conditions of $\Omega(k)$ in Definition 3 are extended with respect to the allowable locations of uncertain elements. Next, the main result is given, which shows that if such developed conditions are satisfied, then system (3.1) is also stabilizable via a linear control.

Here, two sets of matrices $\Omega^l(k)$ and $\Omega^u(k)$ are defined as follows.

Definition 4. Let k be an integer satisfying $0 \le k \le n$. For this k, let $\Omega^l(k) = \{E = (e_{ij}) \in \mathbb{R}^{n\times(n+1)}\}$ be a set of matrices with the following properties:

(i)′ if $1 \le j \le k+1$, then $e_{ij} = 0$ for $j-1 \le i \le 2k-j$;
(ii) if $k+2 \le j \le n+1$, then $e_{ij} = 0$ for $2k-j+1 \le i \le j-1$.

Similarly, let $\Omega^u(k) = \{F = (f_{ij}) \in \mathbb{R}^{n\times(n+1)}\}$ be a set of matrices with the following properties:

(i) if $1 \le j \le k+1$, then $f_{ij} = 0$ for $j-1 \le i \le 2k-j+1$;
(ii)′ if $k+2 \le j \le n+1$, then $f_{ij} = 0$ for $2k-j+2 \le i \le j-1$.

Generally speaking, $\Omega^l(k)$ is obtained from $\Omega(k)$ by replacing condition (i) by (i)′. Figure 1 shows a schematic view of a matrix from $\Omega^l(k)$ for given k. Here, the symbols $*$ and $\triangleleft$ indicate the locations of uncertain elements that are not necessarily equal to zero. In particular, $\triangleleft$ denotes the extended part of allowable uncertainty locations with respect to the lower uncertain part of $\Omega(k)$. Under the conditions of $\Omega(k)$, uncertain elements may take only locations denoted by $*$. For this reason, the allowable uncertainty locations of $\Omega^l(k)$ are more numerous than those of $\Omega(k)$ by the number of locations denoted by $\triangleleft$, where we neglect Δb.

Similarly, note that $\Omega^u(k)$ is obtained from $\Omega(k)$ by replacing condition (ii) by (ii)′. Figure 2 shows a schematic view of the matrix of $\Omega^u(k)$ for given k. Both $*$ and $\triangleright$ indicate the locations of uncertain elements; $\triangleright$ denotes the extended part of allowable uncertainty locations with respect to the upper uncertain part of $\Omega(k)$.

Now we state our main result.

Theorem 1. *Construct a matrix* $\Gamma \in \mathbb{R}^{n\times(n+1)}$ *as*

$$\Gamma = \begin{pmatrix} \Delta A^{30} & \Delta b^0 \end{pmatrix}. \tag{5.1}$$

If, for fixed k,

$$\Gamma \subset \Omega^l(k) \quad \text{or} \quad \Gamma \subset \Omega^u(k), \tag{5.2}$$

then system (3.1) *is stabilizable via a linear control.*

Note that the conditions of Theorem 1 consist not only of ΔA^{30} but also of Δb^0, whereas the previous conditions of Lemma 4 consist only of ΔA^{30}. Moreover, the allowable uncertainty locations of $\Omega^l(k)$ or $\Omega^u(k)$ denoted respectively by $\triangleleft$ or $\triangleright$ are the extended uncertain part from $\Omega(k)$. It is important to note that Γ which satisfies condition (5.2) is equivalent to a basic form of ASC defined in [21]. In this study, it was proved that it is a necessary and sufficient condition for linear time-varying uncertain systems without delays to be quadratically stabilizable using linear state feedback control to have an ASC. Theorem 1 shows that the conditions for the quadratic stabilization of linear uncertain systems without delays are

$$
\begin{array}{c} k+1 \\ \downarrow \end{array}
$$

$$
\begin{pmatrix}
0 & 0 & \ast & \ast & \ast & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast & \ast & 0 & 0 \\
0 & 0 & 0 & \ast & \ast & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdot & 0 & 0 & 0 & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdot & \cdot & 0 & 0 & 0 & \ast & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & \cdot & \cdot & \cdot & 0 & \triangleleft & \ast & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
0 & 0 & \cdot & \cdot & 0 & \triangleleft & \ast & \ast & \ast & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & \cdot & 0 & \triangleleft & \ast & \ast & \ast & \ast & \ast & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \triangleleft & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \triangleleft & \ast & \ast & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast & \ast & \ast & 0 & 0 \\
\triangleleft & \ast & \ast & \ast & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast & \ast & \ast & \ast & 0
\end{pmatrix}
\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \leftarrow k+1 \\ \\ \\ \\ \\ \\ \\ \end{array}
$$

Fig. 1. Matrix $E \in \Omega^l(k)$.

$$
\begin{array}{c} k+1 \\ \downarrow \end{array}
$$

$$
\begin{pmatrix}
0 & 0 & \ast & \ast & \ast & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast & \ast & \triangleright & 0 \\
0 & 0 & 0 & \ast & \ast & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast & \triangleright & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \triangleright & 0 & 0 & 0 & 0 \\
0 & 0 & \cdot & 0 & 0 & 0 & \ast & \ast & \ast & \triangleright & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdot & \cdot & 0 & 0 & 0 & \ast & \triangleright & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \ast & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
0 & 0 & \cdot & \cdot & 0 & 0 & \ast & \ast & \ast & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & \cdot & 0 & 0 & \ast & \ast & \ast & \ast & \ast & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \ast & \ast & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast & \ast & \ast & 0 & 0 \\
0 & \ast & \ast & \ast & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast & \ast & \ast & \ast & 0
\end{pmatrix}
\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \leftarrow k+1 \\ \\ \\ \\ \\ \\ \\ \end{array}
$$

Fig. 2. Matrix $F \in \Omega^u(k)$.

also sufficient conditions for the stabilization of these systems with delays. Consequently, the allowable uncertainty locations for the stabilization of the systems with delays can be increased to those for the quadratic stabilization of systems without delays. The stabilizability conditions are not degraded by the existence of delays that are allowed to take unlimited large values. From this point of view, Theorem 1 is important.

Proof. According to Proposition 4, the existence of T in Eq. (4.23) which assures $P \in \mathcal{M}$ is investigated in the rest of this section. On evaluating the existence of T, it is important how to choose the eigenvalues λ_i $(i = 1, \cdots, n)$ of $(A^0 + bg')$. In fact, the way of constructing T must be changed according to different cases such as $\Gamma \subset \Omega^l(k)$ or $\Gamma \subset \Omega^u(k)$ and $\Delta b = 0$ or $\Delta b \neq 0$.

For the case of $\Gamma \subset \Omega^u(k)$, if $(2k - n) \leq 0$, then $\Delta b = 0$, otherwise $\Delta b \neq 0$. Also, for the case of $\Gamma \subset \Omega^l(k)$, if $(2k - n - 1) \leq 0$, then $\Delta b = 0$, otherwise $\Delta b \neq 0$. Items [I] and [II] indicate the cases of $\Delta b = 0$ and $\Delta b \neq 0$, respectively. Items (i) and (ii) denote the cases of $\Gamma \subset \Omega^u(k)$ and $\Gamma \subset \Omega^l(k)$, respectively.

Here, let μ be a positive number and let α_i $(i = 1, \dots, n)$ be all negative numbers that are different from one another. Let μ be chosen larger than all upper bounds of uncertain elements ΔA^{30} and Δb^0. Let α_i $(i = 1, \dots, n)$ be used for distinguishing eigenvalues from one another.

Proper ways of choosing λ_i $(i = 1, \dots, n)$ are shown below according to four differently classified cases.

[I]-(i) In the case of $\Gamma \subset \Omega^u(k)$ and $(2k - n) \leq 0$,

$$\begin{cases} \lambda_i = \alpha_i \mu^{-1}, & i = 1, \dots, k, \\ \lambda_i = \alpha_i \mu, & i = k+1, \dots, n. \end{cases} \tag{5.3}$$

[I]-(ii) In the case of $\Gamma \subset \Omega^l(k)$ and $(2k - n - 1) \leq 0$,

$$\begin{cases} \lambda_i = \alpha_i \mu^{-1}, & i = 1, \dots, k-1, \\ \lambda_i = \alpha_i \mu, & i = k, \dots, n. \end{cases} \tag{5.4}$$

[II]-(i) In the case of $\Gamma \subset \Omega^u(k)$ and $(2k - n) > 0$,

$$\begin{cases} \lambda_i = \alpha_i \mu^{-2}, & i = 1, \dots, 2k - n + 1, \\ \lambda_i = \alpha_i \mu^{-1}, & i = 2k - n + 2, \dots, k, \\ \lambda_i = \alpha_i \mu, & i = k+1, \dots, n. \end{cases} \tag{5.5}$$

[II]-(ii) In the case of $\Gamma \subset \Omega^l(k)$ and $(2k - n - 1) > 0$,

$$\begin{cases} \lambda_i = \alpha_i \mu^{-3} & i = 1, \dots, 2k - n, \\ \lambda_i = \alpha_i \mu^{-1} & i = 2k - n + 1, \dots, k-1, \\ \lambda_i = \alpha_i \mu & i = k, \dots, n. \end{cases} \tag{5.6}$$

To complete the proof of Theorem 1, we should show that if we choose λ_i $(i = 1, \dots, n)$ as in Eqs. (5.3)–(5.6) for four different cases, then T constructed for such λ_i assures $P \in \mathcal{M}$ for each case.

[I]-(i). Consider the case of $\Gamma \subset \Omega^u(k)$ and $(2k - n) \le 0$.

Suppose that λ_i $(i = 1, \dots, n)$ are chosen as in Eq. (5.3); then it follows that

$$\begin{cases} \lambda_i \in O(-1), & i = 1, \dots, k, \\ \lambda_i \in O(1), & i = k+1, \dots, n. \end{cases} \tag{5.7}$$

In this case, we can write T in Eq. (4.23) as follows:

$$T := \left(\begin{array}{c|c} T^1 & T^2 \end{array} \right) = \left(\begin{array}{c|c} \boxed{\quad 0 \quad} & \boxed{\quad 0 \quad} \\ \vdots & \vdots \\ \boxed{-i+1} & \boxed{i-1} \\ \vdots & \vdots \\ \boxed{-n+1} & \boxed{n-1} \end{array} \right), \tag{5.8}$$

where T^1 and T^2 denote $(n \times k)$- and $(n \times (n-k))$-matrices, respectively. In the above notation, $\boxed{\quad m \quad}$ denotes a row vector, whose entries are functions of μ of order m. For convenience, we adopt such notation for matrices in the subsequent discussion and neglect further explanation when it is clear. The notation of Eq. (5.8) means that all entries of the ith row of T^1 and T^2 are functions of μ of order $(-i+1)$ and $(i-1)$, respectively.

Accordingly, T^{-1} is found to have the following structure:

$$\begin{aligned} T^{-1} &:= \left(\begin{array}{c|c} \hat{T}^{11} & \hat{T}^{12} \\ \hline \hat{T}^{21} & \hat{T}^{22} \end{array} \right) \\ &= \left(\begin{array}{ccccc|ccccc} \boxed{0} & \cdots & \boxed{\begin{matrix} j \\ -1 \end{matrix}} & \cdots & \boxed{\begin{matrix} k \\ -1 \end{matrix}} & \boxed{\begin{matrix} k \\ -2 \end{matrix}} & \cdots & \boxed{\begin{matrix} -j \\ +2k \\ -1 \end{matrix}} & \cdots & \boxed{\begin{matrix} -n \\ +2k \\ -1 \end{matrix}} \\ \hline \boxed{-2k} & \cdots & \boxed{\begin{matrix} j \\ -2k \\ -1 \end{matrix}} & \cdots & \boxed{\begin{matrix} -k \\ -1 \end{matrix}} & \boxed{-k} & \cdots & \boxed{\begin{matrix} -j \\ +1 \end{matrix}} & \cdots & \boxed{\begin{matrix} -n \\ +1 \end{matrix}} \end{array} \right), \end{aligned} \tag{5.9}$$

where $\hat{T}^{11}$, $\hat{T}^{12}$, $\hat{T}^{21}$, and $\hat{T}^{22}$ denote $k \times k$, $k \times (n-k)$, $(n-k) \times k$, and $(n-k) \times (n-k)$ matrices, respectively. The notation $\boxed{m}$ denotes a column vector whose entries are functions of μ of order m. For subsequent discussion, we adopt such notation of matrices without explanation when it is clear. The notation of Eq. (5.9) means that all entries of the jth column of $\hat{T}^{11}$, $\hat{T}^{12}$, $\hat{T}^{21}$, and $\hat{T}^{22}$ are functions of μ that belong respectively to $O(j-1)$, $O(-j+2k-1)$, $O(j-2k-1)$, and $O(-j+1)$.

Considering such structures of T and T^{-1}, it turns out from the careful calculation that $\left|T^{-1}\right| \Delta A^{30} |T|$ is decomposed into four block matrices such as in Eq. (5.9). All entries of each block matrix are functions of μ of the same order. Therefore, $\left|T^{-1}\right| \Delta A^{30} |T|$ has the structure

$$\left|T^{-1}\right| \Delta A^{30} |T| = \left(\begin{array}{c||c} \boxed{-2} & \boxed{2k-1} \\ \hline \boxed{-2k} & \boxed{0} \end{array}\right). \tag{5.10}$$

For the case of $\Gamma \subset \Omega^u(k)$, the inequality $(2k-n) \leq 0$ implies that $\Delta b = 0$. Namely, we obtain $\left|T^{-1}\right| \Delta b^0 |g'| |T| = 0$. Hence P in Eq. (4.32) has the form

$$P = -\Lambda - \left(\begin{array}{c||c} \boxed{-2} & \boxed{2k-1} \\ \hline \boxed{-2k} & \boxed{0} \end{array}\right). \tag{5.11}$$

Taking into account the fact that Λ is a diagonal matrix in which all diagonal entries belong to $O(-1)$ (from the first to kth entry) or $O(1)$ (from the $(k+1)$th to nth entry), we obtain

$$-1 > -2, \quad 1 > 0, \quad -1 > (2k-1) - 1 + (-2k) = -2. \tag{5.12}$$

According to Proposition 2, it is clear from inequalities (5.12) that $P \in \mathcal{M}$.

[I]-(ii). Consider the case $\Gamma \subset \Omega^l(k)$ and $(2k-n-1) \leq 0$.

Assume that λ_i $(i = 1, \ldots, n)$ are chosen as in Eq. (5.4); then

$$\begin{cases} \lambda_i \in O(-1), & i = 1, \ldots, k-1, \\ \lambda_i \in O(1), & i = k, \ldots, n. \end{cases} \tag{5.13}$$

Comparing with case [I]-(i), note that the eigenvalues belonging respectively to $O(-1)$ and $O(1)$ become less and more numerous by one. In this case, we can write T and T^{-1} as follows:

$$T := \left(\, T^1 \mid T^2 \, \right) = \left(\begin{array}{c|c} \boxed{0} & \boxed{0} \\ \vdots & \vdots \\ \boxed{-i+1} & \boxed{i-1} \\ \vdots & \vdots \\ \boxed{-n+1} & \boxed{n-1} \end{array}\right), \tag{5.14}$$

where T^1 and T^2 are $n \times (k-1)$ and $n \times (n-k+1)$ matrices, respectively. Furthermore,

$$
\begin{aligned}
T^{-1} &:= \left(\begin{array}{c|c} \hat{T}^{11} & \hat{T}^{12} \\ \hline \hat{T}^{21} & \hat{T}^{22} \end{array}\right) \\
&= \left(\begin{array}{ccccc||ccccc}
\boxed{0} & \cdots & \boxed{\begin{matrix} j \\ -1 \end{matrix}} & \cdots & \boxed{\begin{matrix} k \\ -2 \end{matrix}} & \boxed{\begin{matrix} k \\ -3 \end{matrix}} & \cdots & \boxed{\begin{matrix} -j \\ +2k \\ -3 \end{matrix}} & \cdots & \boxed{\begin{matrix} -n \\ +2k \\ -3 \end{matrix}} \\ \hline
\boxed{\begin{matrix} -2k \\ +2 \end{matrix}} & \cdots & \boxed{\begin{matrix} j \\ -2k \\ +1 \end{matrix}} & \cdots & \boxed{-k} & \boxed{\begin{matrix} -k \\ +1 \end{matrix}} & \cdots & \boxed{\begin{matrix} -j \\ +1 \end{matrix}} & \cdots & \boxed{\begin{matrix} -n \\ +1 \end{matrix}}
\end{array}\right), \qquad (5.15)
\end{aligned}
$$

where $\hat{T}^{11}$, $\hat{T}^{12}$, $\hat{T}^{21}$, and $\hat{T}^{22}$ are $(k-1) \times (k-1)$, $(k-1) \times (n-k+1)$, $(n-k+1) \times (k-1)$, and $(n-k+1) \times (n-k+1)$ matrices, respectively.

The notation of Eq. (5.14) shows that all entries of the ith row of T^1 and T^2 are functions of μ belonging to $O(-i+1)$ and $O(i-1)$, respectively. The notation of Eq. (5.15) means that all entries of the jth column of $\hat{T}^{11}$, $\hat{T}^{12}$, $\hat{T}^{21}$, and $\hat{T}^{22}$ are functions of μ belonging to $O(j-1)$, $O(-j+2k-3)$, $O(j-2k+1)$, and $O(-j+1)$, respectively.

It turns out from a calculation similar to [I]-(i) that $\left|T^{-1}\right| \Delta A^{30} |T|$ is also decomposed into four block matrices such as Eq. (5.15). All entries of each block matrix are functions of μ of the same order.

For the case of $\Gamma \subset \Omega^l(k)$, the inequality $(2k-n-1) \leq 0$ implies that $\Delta b = 0$. From $\left|T^{-1}\right| \Delta b^0 |g'| |T| = 0$ we find that P has the structure

$$
P = -\Lambda - \left(\begin{array}{c||c} \boxed{-2} & \boxed{2k-2} \\ \hline \boxed{-2k+1} & \boxed{0} \end{array}\right). \qquad (5.16)
$$

Taking into account the fact that Λ is a diagonal matrix such that all diagonal entries belong to $O(-1)$ (from the first to $(k-1)$th entry) or $O(1)$ (from the kth to nth entry), we obtain

$$
-1 > -2, \quad 1 > 0, \quad -1 > (2k-2) - 1 + (-2k+1) = -2. \qquad (5.17)
$$

According to Proposition 2, the inequalities (5.17) show that $P \in \mathcal{M}$.

[II]-(i). Consider the case $\Gamma \subset \Omega^u(k)$ and $(2k-n) > 0$.

In this case, the inequality $(2k-n) > 0$ implies $\Delta b \neq 0$. Hence not only $\left|T^{-1}\right| \Delta A^{30} |T|$, but also $\left|T^{-1}\right| \Delta b^0 |g'| |T|$ should be considered. Assume

$$
\begin{array}{c}
\hspace{4em} k+1 \\
\hspace{4em} \downarrow
\end{array}
$$

$$
\left(\begin{array}{ccccccccccccc}
0 & 0 & \ast & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast & \ast \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & \ast & \ast & \cdot & \cdot & \cdot & \cdot & \cdot & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & \ast & \ast & \cdot & \cdot & \cdot & \ast & \ast & \triangleright \\
0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \triangleright & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \triangleright & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \triangleright & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0
\end{array}\right)
\begin{array}{l}
\\ \\ \\ \\
\leftarrow 2k-n+1 \\
\\ \\ \\
\leftarrow k+1 \\
\\ \\ \\
\end{array}
$$

$$
\begin{array}{c}
\uparrow \\
2k-n+1
\end{array}
$$

Fig. 3. Matrix $F \in \Omega^u(k)$ in the case $(2k-n) > 0$.

that λ_i $(i = 1, \dots, n)$ are chosen as in Eq. (5.5); then

$$
\begin{cases}
\lambda_i \in O(-2), & i = 1, \dots, 2k-n+1, \\
\lambda_i \in O(-1), & i = 2k-n+2, \dots, k, \\
\lambda_i \in O(1), & i = k+1, \dots, n.
\end{cases} \tag{5.18}
$$

A comparison with case [I]-(i) shows that the eigenvalues of $O(-1)$ are further decomposed into those of $O(-1)$ and $O(-2)$. In addition, the number of eigenvalues having order -2 is equal to the number of entries which must be zero in the lowest row of ΔA^{30}, as is shown in Fig. 3.

In this case, we can write T as follows:

$$
\begin{aligned}
T &:= \left(\ T^1 \mid T^2 \mid T^3 \ \right) \\
&= \left(\begin{array}{c|c|c}
\boxed{0} & \boxed{0} & \boxed{0} \\
\vdots & \vdots & \vdots \\
\boxed{-2i+2} & \boxed{-i+1} & \boxed{i-1} \\
\vdots & \vdots & \vdots \\
\boxed{-2n+2} & \boxed{-n+1} & \boxed{n-1}
\end{array} \right),
\end{aligned} \tag{5.19}
$$

where T^1, T^2, and T^3 denote $n \times (2k-n+1)$, $n \times (n-k-1)$, and $n \times (n-k)$ matrices, respectively.

Accordingly, T^{-1} has the following structure:

$$
\begin{aligned}
T^{-1} &:= \left(\begin{array}{c|c|c} \hat{T}^{11} & \hat{T}^{12} & \hat{T}^{13} \\ \hline \hat{T}^{21} & \hat{T}^{22} & \hat{T}^{23} \\ \hline \hat{T}^{31} & \hat{T}^{32} & \hat{T}^{33} \end{array}\right) \\
&= \left(\begin{array}{ccccc||ccccc||ccccc}
\boxed{0} & \cdots & \boxed{\begin{smallmatrix} 2j \\ -2 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} -2n \\ +4k \end{smallmatrix}} & \boxed{\begin{smallmatrix} -2n \\ +4k \\ +1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} -n \\ +2k \\ +j \\ -1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} -n \\ +3k \\ -1 \end{smallmatrix}} & \boxed{\begin{smallmatrix} -n \\ +3k \\ -2 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} -n \\ +4k \\ -j \\ -1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} -2n \\ +4k \\ -1 \end{smallmatrix}} \\ \hline
\boxed{\begin{smallmatrix} n \\ -2k \\ -1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} n \\ -2k \\ +2j \\ -3 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} -n \\ +2k \\ -1 \end{smallmatrix}} & \boxed{\begin{smallmatrix} -n \\ +2k \\ +1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} j \\ -1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} k \\ -1 \end{smallmatrix}} & \boxed{\begin{smallmatrix} k \\ -2 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} 2k \\ -j \\ -1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} -n \\ +2k \\ -1 \end{smallmatrix}} \\ \hline
\boxed{\begin{smallmatrix} n \\ -4k \\ -1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} n \\ -4k \\ +2j \\ -3 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} -n \\ -1 \end{smallmatrix}} & \boxed{\begin{smallmatrix} -n \\ +1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} -2k \\ +j \\ -1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} -k \\ -1 \end{smallmatrix}} & \boxed{-k} & \cdots & \boxed{\begin{smallmatrix} -j \\ +1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} -n \\ +1 \end{smallmatrix}}
\end{array}\right).
\end{aligned}
\tag{5.20}
$$

In all these (3×3)-block matrices, all blocks of the first, second, and third row represent matrices with $(2k - n + 1)$, $(n - k - 1)$, and $(n - k)$ rows, respectively, and all blocks of the first, second, and third column represent matrices with $(2k - n + 1)$, $(n - k - 1)$, and $(n - k)$ columns, respectively.

The notation of Eq. (5.19) means that all entries of the ith row of T^1, T^2, and T^3 denote functions of μ belonging to $O(-2i + 2)$, $O(-i + 1)$, and $O(i - 1)$, respectively.

The notation of Eq. (5.20) means that all entries of the jth column of $\hat{T}^{11}$, $\hat{T}^{12}$, $\hat{T}^{13}$, $\hat{T}^{21}$, $\hat{T}^{22}$, $\hat{T}^{23}$, and $\hat{T}^{31}$, $\hat{T}^{32}$, $\hat{T}^{33}$ are functions of μ belonging to $O(2j - 2)$, $O(-n + 2k + j - 1)$, $O(-n + 4k - j - 1)$, $O(n - 2k + 2j - 3)$, $O(j - 1)$, $O(2k - j - 1)$, and $O(n - 4k + 2j - 3)$, $O(-2k + j - 1)$, $O(-j + 1)$, respectively.

From the relations between the roots and the coefficients of the characteristic equation $\det(A^0 + bg')$, we find that g' has the following structure:

$$
\begin{aligned}
g' &:= \left(\begin{array}{c|c|c} G^1 & G^2 & G^3 \end{array}\right) \\
&= \left(\begin{array}{ccccc||ccccc||ccccc}
\boxed{\begin{smallmatrix} 2n \\ -4k \\ -1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} 2n \\ -4k \\ +2j \\ -3 \end{smallmatrix}} & \cdots & \boxed{-1} & \boxed{1} & \cdots & \boxed{\begin{smallmatrix} n \\ -2k \\ +j \\ -1 \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} n \\ -k \\ -1 \end{smallmatrix}} & \boxed{\begin{smallmatrix} n \\ -k \end{smallmatrix}} & \cdots & \boxed{\begin{smallmatrix} n \\ -j \\ +1 \end{smallmatrix}} & \cdots & \boxed{1}
\end{array}\right),
\end{aligned}
\tag{5.21}
$$

where G^1, G^2, and G^3 represent $1 \times (2k - n + 1)$, $1 \times (n - k - 1)$, and $1 \times (n - k)$ matrices, respectively.

The notation of Eq. (5.21) means that the entry of the jth column of G^1, G^2, and G^3 is a scalar function of μ belonging to $O(2n-4k+2j-3)$, $O(n-2k+j-1)$, and $O(n-j+1)$, respectively.

Careful calculation shows that $\left|T^{-1}\right| \Delta b^0 \left|g'\right| \left|T\right|$ and $\left|T^{-1}\right| \Delta A^{30} \left|T\right|$ are decomposed into the same structure as nine block matrices in Eq. (5.20). All entries of each block matrix are functions of μ of the same order. Considering such structures and properties, we obtain the following:

$$\left|T^{-1}\right| \Delta b^0 \left|g'\right| \left|T\right| = \left(\begin{array}{c|c|c} \boxed{-3} & \boxed{-n+2k-2} & \boxed{-n+4k-2} \\ \hline \boxed{n-2k-4} & \boxed{-3} & \boxed{2k-3} \\ \hline \boxed{n-4k-4} & \boxed{-2k-3} & \boxed{-3} \end{array}\right), \tag{5.22}$$

$$\left|T^{-1}\right| \Delta A^{30} \left|T\right| = \left(\begin{array}{c|c|c} \boxed{-4} & \boxed{-n+2k-2} & \boxed{-n+4k-1} \\ \hline \boxed{n-2k-5} & \boxed{-2} & \boxed{2k-1} \\ \hline \boxed{n-4k-1} & \boxed{-2k} & \boxed{0} \end{array}\right). \tag{5.23}$$

By comparing Eq. (5.22) with Eq. (5.23), we obtain

$$\begin{aligned} P &= -\Lambda - \left|T^{-1}\right| \Delta A^{30} \left|T\right| - \left|T^{-1}\right| \Delta b^0 \left|g'\right| \left|T\right| \\ &= -\Lambda - \left(\begin{array}{c|c|c} \boxed{-3} & \boxed{-n+2k-2} & \boxed{-n+4k-1} \\ \hline \boxed{n-2k-4} & \boxed{-2} & \boxed{2k-1} \\ \hline \boxed{n-4k-1} & \boxed{-2k} & \boxed{0} \end{array}\right). \end{aligned} \tag{5.24}$$

Taking into account the fact that Λ is a diagonal matrix such that all diagonal entries belong to $O(-2)$ (from the first to $(2k-n+1)$th entry) or

$O(-1)$ (from the $(2k-n+2)$th to kth entry) or $O(1)$ (from the $(k+1)$th to nth entry), we obtain

$$\begin{gathered} -2 > -3, \quad -1 > -2, \quad 1 > 0, \\ -2 > (-n+2k-2) - (-1) + (n-2k-4) = -5, \\ -2 > (-n+4k-1) - 1 + (n-4k-1) = -3, \\ -1 > (2k-1) - 1 + (-2k) = -2, \\ -2 > (-n+2k-2) - (-1) + (n-2k-3) = -4. \end{gathered} \tag{5.25}$$

According to Proposition 3, inequalities (5.25) imply $P \in \mathcal{M}$.

[II]-(ii). Consider the case $\Gamma \subset \Omega^l(k)$ and $(2k-n-1) > 0$.

In this case, the inequality $(2k-n-1) > 0$ implies $\Delta b \neq 0$. Then, not only $\left|T^{-1}\right| \Delta A^{30} \left|T\right|$, but also $\left|T^{-1}\right| \Delta b^0 \left|g'\right| \left|T\right|$ should be considered. Suppose that λ_i $(i = 1, \cdots, n)$ are chosen as in Eq. (5.6); then it follows that

$$\begin{cases} \lambda_i \in O(-3), & i = 1, \ldots, 2k-n, \\ \lambda_i \in O(-1), & i = 2k-n+1, \ldots, k-1, \\ \lambda_i \in O(1), & i = k, \ldots, n. \end{cases} \tag{5.26}$$

A comparison with case [I]-(ii) shows that the eigenvalues of $O(-1)$ are further decomposed into those of $O(-1)$ and $O(-3)$. It is worth to note that if we choose eigenvalues belonging to $O(-2)$ instead of $O(-3)$, then we fail in our proof. In addition, note that the eigenvalues having order -3 are numerically equal to the entries which must be zero in the lowest row of ΔA^{30}, as is shown in Fig. 4.

In this case, we can write T as follows:

$$T := \left(\; T^1 \; \middle| \; T^2 \; \middle| \; T^3 \; \right) = \left(\begin{array}{c|c|c} \boxed{0} & \boxed{0} & \boxed{0} \\ \vdots & \vdots & \vdots \\ \boxed{-3i+3} & \boxed{-i+1} & \boxed{i-1} \\ \vdots & \vdots & \vdots \\ \boxed{-3n+3} & \boxed{-n+1} & \boxed{n-1} \end{array} \right), \tag{5.27}$$

where T^1, T^2, and T^3 denote $n \times (2k-n)$, $n \times (n-k-1)$, and $n \times (n-k+1)$ matrices, respectively.

The notation of Eq. (5.27) means that all entries of the ith row of T^1, T^2, and T^3 are functions of μ belonging to $O(-3i+3)$, $O(-i+1)$, and $O(i-1)$, respectively.

$$
\begin{array}{c}
 k+1 \\
 \downarrow
\end{array}
$$

$$
\begin{pmatrix}
0 & 0 & * & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & * & * & \cdot & \cdot & \cdot & \cdot & \cdot & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & \cdot & \cdot & \cdot & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \triangleleft & * & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & \triangleleft & * & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & \triangleleft & * & * & * & * & * & * & * & 0
\end{pmatrix}
\begin{array}{l}
\\ \\ \\ \leftarrow 2k-n \\ \\ \\ \\ \\ \leftarrow k+1 \\ \\ \\ \\
\end{array}
$$

$$
\begin{array}{c}
 \uparrow \\
 2k-n
\end{array}
$$

Fig. 4. Matrix $E \in \Omega^l(k)$ in the case $(2k-n-1) > 0$.

Accordingly, T^{-1} has the structure

$$
T^{-1} := \left(\begin{array}{c|c|c}
\hat{T}^{11} & \hat{T}^{12} & \hat{T}^{13} \\ \hline
\hat{T}^{21} & \hat{T}^{22} & \hat{T}^{23} \\ \hline
\hat{T}^{31} & \hat{T}^{32} & \hat{T}^{33}
\end{array}\right)
$$

$$
= \left(\begin{array}{ccccc||ccccc||ccccc}
\boxed{0} & \cdots & \boxed{\begin{matrix}3j\\-3\end{matrix}} & \cdots & \boxed{\begin{matrix}-3n\\+6k\\-3\end{matrix}} & \boxed{\begin{matrix}-3n\\+6k\\-2\end{matrix}} & \cdots & \boxed{\begin{matrix}-2n\\+4k\\+j\\-3\end{matrix}} & \cdots & \boxed{\begin{matrix}-2n\\+5k\\-4\end{matrix}} & \boxed{\begin{matrix}-2n\\+5k\\-5\end{matrix}} & \cdots & \boxed{\begin{matrix}-2n\\+6k\\-j\\-5\end{matrix}} & \cdots & \boxed{\begin{matrix}-3n\\+6k\\-5\end{matrix}} \\ \hline
\boxed{\begin{matrix}2n\\-4k\end{matrix}} & \cdots & \boxed{\begin{matrix}2n\\-4k\\+3j\\-3\end{matrix}} & \cdots & \boxed{\begin{matrix}-n\\+2k\\-3\end{matrix}} & \boxed{\begin{matrix}-n\\+2k\end{matrix}} & \cdots & \boxed{\begin{matrix}j\\-1\end{matrix}} & \cdots & \boxed{\begin{matrix}k\\-2\end{matrix}} & \boxed{\begin{matrix}k\\-3\end{matrix}} & \cdots & \boxed{\begin{matrix}2k\\-j\\-3\end{matrix}} & \cdots & \boxed{\begin{matrix}-n\\+2k\\-3\end{matrix}} \\ \hline
\boxed{\begin{matrix}2n\\-6k\\+2\end{matrix}} & \cdots & \boxed{\begin{matrix}2n\\-6k\\+3j\\-1\end{matrix}} & \cdots & \boxed{\begin{matrix}-n\\-1\end{matrix}} & \boxed{\begin{matrix}-n\\+2\end{matrix}} & \cdots & \boxed{\begin{matrix}-2k\\+j\\+1\end{matrix}} & \cdots & \boxed{-k} & \boxed{\begin{matrix}-k\\+1\end{matrix}} & \cdots & \boxed{\begin{matrix}-j\\+1\end{matrix}} & \cdots & \boxed{\begin{matrix}-n\\+1\end{matrix}}
\end{array}\right). \tag{5.28}
$$

In all these 3×3 block matrices, all blocks of the first, second, and third row represent matrices with $(2k-n)$, $(n-k-1)$, and $(n-k+1)$ rows, respectively, and all blocks of the first, second, and third column represent matrices with $(2k-n)$, $(n-k-1)$, and $(n-k+1)$ columns, respectively.

The notation of Eq. (5.28) means that all entries of the jth column of $\hat{T}^{11}$, $\hat{T}^{12}$, $\hat{T}^{13}$, $\hat{T}^{21}$, $\hat{T}^{22}$, $\hat{T}^{23}$, and $\hat{T}^{31}$, $\hat{T}^{32}$, $\hat{T}^{33}$ are functions of μ belonging to $O(3j-3)$, $O(-2n+4k+j-3)$, $O(-2n+6k-j-5)$, $O(2n-4k+3j-3)$, $O(j-1)$, $O(2k-j-3)$, and $O(2n-6k+3j-1)$, $O(-2k+j+1)$, $O(-j+1)$, respectively.

From the relations between the roots and coefficients of the characteristic equation $\det(A^0 + bg')$ we find that g' has the structure

$$
\begin{aligned}
& g' := \left(\; G^1 \;\middle|\; G^2 \;\middle|\; G^3 \; \right) \\
&= \left(\boxed{\begin{matrix} 3n \\ -6k \\ +2 \end{matrix}} \cdots \boxed{\begin{matrix} 3n \\ -6k \\ +3j \\ -1 \end{matrix}} \cdots \boxed{-1} \;\middle\|\; \boxed{2} \cdots \boxed{\begin{matrix} n \\ -2k \\ +j \\ +1 \end{matrix}} \cdots \boxed{\begin{matrix} n \\ -k \end{matrix}} \;\middle\|\; \boxed{\begin{matrix} n \\ -k \\ +1 \end{matrix}} \cdots \boxed{\begin{matrix} n \\ -j \\ +1 \end{matrix}} \cdots \boxed{1} \right),
\end{aligned}
\tag{5.29}
$$

where G^1, G^2, and G^3 denote $1 \times (2k-n)$, $1 \times (n-k-1)$, and $1 \times (n-k+1)$ matrices, respectively.

The notation of Eq. (5.29) means that the entry of the jth column of G^1, G^2, and G^3 is a scalar function of μ belonging to $O(3n-6k+3j-1)$, $O(n-2k+j+1)$, and $O(n-j+1)$, respectively.

A calculation similar to that of [II]-(i) shows that $\left|T^{-1}\right| \Delta b^0 \left|g'\right| \left|T\right|$ and $\left|T^{-1}\right| \Delta A^{30} \left|T\right|$ are decomposed into the same structure as nine block matrices in Eq. (5.28). All entries of each block matrix are functions of μ of the same order. Considering such structures and properties, we obtain

$$
\left|T^{-1}\right| \Delta b^0 \left|g'\right| \left|T\right| = \left(\begin{array}{c|c|c} \boxed{-4} & \boxed{-2n+4k-4} & \boxed{-2n+6k-6} \\ \hline \boxed{2n-4k-4} & \boxed{-4} & \boxed{2k-6} \\ \hline \boxed{2n-6k-2} & \boxed{-2k-2} & \boxed{-4} \end{array} \right),
\tag{5.30}
$$

$$
\left|T^{-1}\right| \Delta A^{30} \left|T\right| = \left(\begin{array}{c|c|c} \boxed{-5} & \boxed{-2n+4k-4} & \boxed{-2n+6k-4} \\ \hline \boxed{2n-4k-3} & \boxed{-2} & \boxed{2k-2} \\ \hline \boxed{2n-6k+1} & \boxed{-2k} & \boxed{0} \end{array} \right).
\tag{5.31}
$$

By comparing Eq. (5.30) with Eq. (5.31), we obtain

$$P = -\Lambda - \left|T^{-1}\right| \Delta A^{30} \left|T\right| - \left|T^{-1}\right| \Delta b^{0} \left|g'\right| \left|T\right|$$
$$= -\Lambda - \left(\begin{array}{c|c|c} \boxed{-4} & \boxed{-2n+4k-4} & \boxed{-2n+6k-4} \\ \hline \boxed{2n-4k-3} & \boxed{-2} & \boxed{2k-2} \\ \hline \boxed{2n-6k+1} & \boxed{-2k} & \boxed{0} \end{array}\right). \quad (5.32)$$

Taking into account the fact that Λ is a diagonal matrix such that all diagonal entries belong to $O(-3)$ (from the first to $(2k-n)$th entry) or $O(-1)$ (from the $(2k-n+1)$th to $(k-1)$th entry) or $O(1)$ (kth to nth entry), we obtain

$$\begin{gathered} -3 > -4, \quad -1 > -2, \quad 1 > 0, \\ -3 > (-2n+4k-4) - (-1) + (2n-4k-3) = -6, \\ -3 > (-2n+6k-4) - 1 + (2n-6k+1) = -4, \\ -1 > (2k-2) - 1 + (-2k) = -3, \\ -3 > (-2n+4k-4) - (-1) + (2n-4k-2) = -5. \end{gathered} \quad (5.33)$$

According to Proposition 3, it is clear from inequalities (5.33) that $P \in \mathcal{M}$.

The proof of [I]-(i), [I]-(ii), [II]-(i), and [II]-(ii) is completed. □

6. Algorithm

Here, the construction of the controller applying the main theorem is briefly shown. The given system must be controllable and, therefore, it should be transformed into a system whose nominal system has a controllable canonical form satisfying Assumption 1.

Step 1. It is verified whether the uncertain entries of the system matrices satisfy condition (5.2) in Theorem 1.

Step 2. The value of k is clear legible from the structure of Γ, and the system can be classified into [I]-(i), [I]-(ii), [II]-(i), or [II]-(ii) when the system satisfies condition (5.2).

Step 3. Classification of the system leads us to determine the proper means of choosing λ_i $(i = 1, \ldots, n)$ from Eqs. (5.3)–(5.6).

Step 4. Each α_i $(i = 1, \ldots, n)$ can be arbitrarily chosen as a negative number different from the others. Since α_i should belong to $O(0)$ of μ, the absolute value of α_i is chosen to be as much as the value of each entry of ΔA^{30} and Δb^0.

Step 5. We take μ as a positive number much larger than the value of every entry of ΔA^{30} and Δb^0.

Step 6. After setting α_i and μ, we can obtain the proper λ_i from Eqs. (5.3)–(5.6). Vector g can be determined from the relations between the roots and coefficients of the characteristic equation $\det(A^0 + bg')$.

Step 7. For the constructed T and g, we verify whether P in Eq. (4.32) satisfies $P \in \mathcal{M}$. If $P \in \mathcal{M}$ is satisfied, then the process can proceed; otherwise, we must return to Step 5 and change μ into a larger value than the present one. (Sufficiently large μ such that $P \in \mathcal{M}$ always exists however large the given upper bounds of uncertain parameters might be.)

Remark 1. Taking into account the cost performance, we should repeat Steps 5–7 to find the smallest possible μ satisfying $P \in \mathcal{M}$.

7. Example

An illustrative example is given here. Consider the following system:

$$\dot{x}(t) = A^0x(t) + \Delta A^1(t)x(t) + \Delta A^{21}(t)x(t - \tau_1(t)) + (b + \Delta b(t))u(t), \quad (7.1)$$

where

$$A^0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad b + \Delta b(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} (\sin t)/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta A^1(t) = \begin{bmatrix} 0 & 0 & (\sin t)/4 & (\sin t)/4 & (\sin t)/4 \\ 0 & 0 & 0 & (\sin t)/4 & (\sin t)/4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\cos t)/4 & 0 \\ 0 & 0 & (\cos t)/4 & (\cos t)/4 & (\cos t)/4 \end{bmatrix},$$

$$\Delta A^{21}(t) = \begin{bmatrix} 0 & 0 & (\sin t)/4 & (\sin t)/4 & (\sin t)/4 \\ 0 & 0 & 0 & (\sin t)/4 & (\sin t)/4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\cos t)/4 & 0 \\ 0 & 0 & (\cos t)/4 & (\cos t)/4 & (\cos t)/4 \end{bmatrix},$$

where $\tau_1(t) = 0.6(21 + \sin t)$. The subsequent discussion follows the algorithm provided in Sec. 6.

Step 1. For the above system, the locations of the time-varying system parameters satisfy condition (5.2) of Theorem 1. Consequently, we see that system (7.1) is stabilizable via a linear control.

Step 2. We find that k is equal to 3 from the structure of Γ as follows:

$$\Gamma = \begin{pmatrix} \Delta A^{30} & \Delta b^0 \end{pmatrix} \subset \Omega^u(3) = \begin{bmatrix} 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & * & * & 0 \end{bmatrix}. \tag{7.2}$$

System (7.1) is classified to the case $\Gamma \subset \Omega^u(k)$ and $(2k-n) > 0$.

Step 3. The proper way of choosing λ_i $(i = 1, \ldots, n)$ is determined by Eq. (5.5) since system (7.1) belongs to case of [II]-(i).

Step 4. Note that the upper bounds of the time-varying system parameters are given as follows:

$$\Delta A^{30} = \Delta A^{10} + \Delta A^{210} = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0.5 \end{bmatrix}, \quad \Delta b^0 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then, α_i $(i = 1, \ldots, n)$ are set as negative numbers as follows:

$$\alpha_1 = -1.0,\ \alpha_2 = -0.3,\ \alpha_3 = -0.9,\ \alpha_4 = -0.5,\ \alpha_5 = -1.3. \tag{7.3}$$

Step 5. μ is chosen as a positive number that is larger than all upper bounds of uncertain parameters:

$$\mu = 11. \tag{7.4}$$

Step 6. We obtain the following eigenvalues:

$$\begin{aligned} \lambda_1 = -8.2645 \times 10^{-3},\ \lambda_2 = -2.4793 \times 10^{-3}, \\ \lambda_3 = -8.1818 \times 10^{-2},\ \lambda_4 = -5.5,\ \lambda_5 = -14.3. \end{aligned} \tag{7.5}$$

The vector g can be found from the relations between the eigenvalues and coefficients of the characteristic equation $\det(A^0 + bg')$. For the eigenvalues from Eq. (7.5), g' is given as follows:

$$g' = \begin{bmatrix} -1.3186 \times 10^{-4} & -7.0781 \times 10^{-2} & -7.2978 & -80.484 & -19.893 \end{bmatrix}. \tag{7.6}$$

Step 7. Let T in Eq. (4.23) be constructed by $\lambda_1, \ldots, \lambda_5$ in Eq. (7.5). Now, we can calculate $\left|T^{-1}\right| \Delta A^{30} \left|T\right|$ and $\left|T^{-1}\right| \Delta b^{0} \left|g'\right| \left|T\right|$ as follows:

$$\left|T^{-1}\right| \Delta A^{30} \left|T\right| = \begin{bmatrix} 1.2721\times10^{-3} & 9.9785\times10^{-5} & 3.3273\times10^{-1} & 1.7385\times10^{5} & 5.9977\times10^{6} \\ 1.2169\times10^{-3} & 9.5654\times10^{-5} & 3.1564\times10^{-1} & 1.6848\times10^{5} & 5.8637\times10^{6} \\ 8.9693\times10^{-5} & 7.215\times10^{-6} & 2.0742\times10^{-2} & 5.9211\times10^{3} & 1.5657\times10^{5} \\ 2.6747\times10^{-8} & 2.2168\times10^{-9} & 5.2718\times10^{-6} & 1.2227 & 3.043\times10^{1} \\ 1.4087\times10^{-9} & 1.2219\times10^{-10} & 2.0247\times10^{-7} & 4.0043\times10^{-2} & 1.2031 \end{bmatrix} \quad (7.7)$$

$$\left|T^{-1}\right| \Delta b^{0} \left|g'\right| \left|T\right| = \begin{bmatrix} 3.0115\times10^{-4} & 8.4421\times10^{-5} & 2.3826\times10^{-2} & 7.5992\times10^{3} & 2.5526\times10^{5} \\ 9.2929\times10^{-4} & 2.605\times10^{-4} & 7.3522\times10^{-2} & 2.3449\times10^{4} & 7.8768\times10^{5} \\ 2.2598\times10^{-6} & 6.3349\times10^{-7} & 1.7879\times10^{-4} & 5.7024\times10^{1} & 1.9155\times10^{3} \\ 1.0499\times10^{-11} & 2.9431\times10^{-12} & 8.3062\times10^{-10} & 2.6492\times10^{-4} & 8.899\times10^{-3} \\ 2.2736\times10^{-13} & 6.3734\times10^{-14} & 1.7987\times10^{-11} & 5.737\times10^{-6} & 1.9271\times10^{-4} \end{bmatrix}. \quad (7.8)$$

Considering $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$, we obtain

$$P = -\Lambda - \left|T^{-1}\right| \Delta A^{30} \left|T\right| - \left|T^{-1}\right| \Delta b^{0} \left|g'\right| \left|T\right| = \begin{bmatrix} 6.6912\times10^{-3} & -1.8421\times10^{-4} & -3.5656\times10^{-1} & -1.8144\times10^{5} & -6.253\times10^{6} \\ -2.1462\times10^{-3} & 2.1232\times10^{-3} & -3.8916\times10^{-1} & -1.9193\times10^{5} & -6.6513\times10^{6} \\ -9.1953\times10^{-5} & -7.8485\times10^{-6} & 6.0897\times10^{-2} & -5.9781\times10^{3} & -1.5848\times10^{5} \\ -2.6757\times10^{-8} & -2.2197\times10^{-9} & -5.2726\times10^{-6} & 4.277 & -3.0439\times10^{1} \\ -1.4089\times10^{-9} & -1.2225\times10^{-10} & -2.0249\times10^{-7} & -4.0049\times10^{-2} & 1.3097\times10^{1} \end{bmatrix}, \quad (7.9)$$

$$P^{-1} = \begin{bmatrix} 1.1225\times 10^{4} & 3.6201\times 10^{3} & 2.2216\times 10^{5} & 1.0649\times 10^{9} & 1.2361\times 10^{10} \\ 4.4499\times 10^{4} & 1.4931\times 10^{4} & 8.8987\times 10^{5} & 4.2652\times 10^{9} & 4.951\times 10^{10} \\ 4.7027\times 10^{1} & 1.5321\times 10^{1} & 9.5336\times 10^{2} & 4.5042\times 10^{6} & 5.2239\times 10^{7} \\ 1.7175\times 10^{-4} & 5.5951\times 10^{-5} & 3.4358\times 10^{-3} & 1.6618\times 10^{1} & 1.9062\times 10^{2} \\ 2.8752\times 10^{-6} & 9.3679\times 10^{-7} & 5.7452\times 10^{-5} & 2.7483\times 10^{-1} & 3.2588 \end{bmatrix} \geq 0. \quad (7.10)$$

From Eqs. (7.9) and (7.10), it is obvious that all off-diagonal entries of P are nonpositive and that P^{-1} is nonnegative. Then, it follows that $P \in \mathcal{M}$. Therefore, system (7.1) is stabilizable by constructing the feedback gain g in Eq. (7.6). In designing a stabilizing feedback gain, the upper bounds of delays are assumed not to be known. For this reason, the controller designed from the given upper bounds of uncertain parameters is applicable for any delays, however large they might be since the stability condition used here is independent of time-varying delays.

For a class of linear time-varying uncertain delay systems, the LMI condition derived from the Lyapunov–Krasovskii functional approach is a commonly used tool. In the sequel, we illustrate the result obtained by the LMI condition [14] for system (7.1) by an example.

In most papers using the Lyapunov–Krasovskii approach, the uncertain parameters are assumed to satisfy the following condition:

$$\begin{bmatrix} \Delta A^{1}(t) & \Delta A^{21}(t) & \Delta b(t) \end{bmatrix} = H\Delta(t) \begin{bmatrix} A^{10} & A^{20} & b^{0} \end{bmatrix}, \quad (7.11)$$

where H, A^{10}, A^{20}, and b^0 are constant matrices such that $\Delta'(t)\Delta(t) \leq I$. Then, we set them as follows:

$$H = I, \quad \Delta(t) = \begin{bmatrix} \sin t & 0 & 0 & 0 & 0 \\ 0 & \sin t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos t & 0 \\ 0 & 0 & 0 & 0 & \cos t \end{bmatrix},$$

$$A^{10} = A^{20} = \begin{bmatrix} 0 & 0 & \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}, \quad b^{0} = \begin{bmatrix} 2\varepsilon \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where $\varepsilon \in \mathbb{R}$ is the upper bound of uncertain parameters.

The paper [14] shows that system (7.1) can be stabilized if there exist symmetric positive-definite matrices X and Z, a matrix Y, and a positive scalar γ such that

$$\begin{bmatrix} \Sigma_{11} & 0 & \Sigma_{13} \\ 0 & -(1-\tau_d)Z & XA^{20'} \\ \Sigma_{13}' & A^{20}X & -\gamma I \end{bmatrix} < 0, \tag{7.12}$$

where

$$\begin{aligned} \Sigma_{11} &= A^0X + XA^{0'} + bY + Y'b' + Z + \gamma HH', \\ \Sigma_{13} &= XA^{10'} + Y'E_b'; \end{aligned} \tag{7.13}$$

τ_d is assumed to be a positive constant such that $\dot{\tau}(t) \le \tau_d < 1$. For system (7.1), $\tau_d = 0.6$, although the method proposed here is free from the restriction on $\dot{\tau}(t)$.

For system (7.1), the value of ε should be taken as 0.25. However, if we choose $\varepsilon = 0.25$, then the LMI solver on the computer calculation cannot identify the solution. Checking the feasibility on the LMI solver repeatedly with ε decreasing, we can identify the maximum value as $\varepsilon = 0.145$. Using the obtained solution, the stabilizing feedback gain is given as

$$g' = YX^{-1} = \begin{bmatrix} -2.3787 & -3013 & -8244 & -10608 & -5726 \end{bmatrix}. \tag{7.14}$$

As is mentioned in the Introduction, the stabilizability conditions with the LMI approach belong to the first category and depend on the ranges of uncertain parameters. There is usually no need for redesigning the controller when the uncertain parameters are greater than the certain amount values. On the contrary, the stabilizability condition given here is independent of the ranges of uncertain parameters. We can redesign the controller for the improvement of the robustness simply by modifying the design parameter μ when the uncertain parameters exceed the upper bounds given above.

Comparing Eq. (7.6) with Eq. (7.14), we see that the stabilizing feedback gain obtained using the proposed method is much less than that obtained using the LMI approach.

8. Conclusion

The stabilization problem of linear time-varying uncertain delay systems using linear memoryless state feedback control was considered in this paper. In particular, we investigated the permissible locations of uncertain elements and delays, both of which are allowed to take unlimited large values for the stabilization. The stabilizability conditions provided here are independent of the ranges of the uncertain parameters and delays; they can be easily verified merely by examining the uncertainty locations in given system matrices. In this paper, the obtained conditions were developed; the results showed that the conditions for the quadratic stabilization of linear uncertain systems without delays are also sufficient conditions for the stabilization of these systems with delays. For this reason, the allowable uncertainty locations for the stabilization of linear uncertain systems with delays can be increased numerically with respect to those for the quadratic stabilization of these systems without delays. From this point of view, we found that the stabilizability conditions are not degraded by the existence of time-varying delays, which has an important meaning. For systems satisfying the stabilizability conditions, a simple control design procedure was also presented and illustrated by an example.

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