Controllability and Observability of Linear Time-Invariant Uncertain Systems Irrespective of Bounds of Uncertain Parameters

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***Abstract*—In this paper, the controllability and observability of linear time-invariant uncertain systems are investigated. The systems under consideration contain time-invariant uncertain parameters that may take arbitrarily large values. In such a situation, the locations of uncertain parameters in system matrices play an important role. We examine the permissible locations of uncertain parameters in system matrices for a linear uncertain system to be controllable and observable independently of the bounds of the uncertain parameters. The objective of this paper is to show that a linear uncertain system is controllable and observable, irrespective of the bounds of uncertain parameters, if and only if the system has a particular configuration called a complete generalized antisymmetric stepwise configuration (CGASC). Furthermore, the dual configuration of a CGASC is introduced and studied here.**

***Index Terms*—Controllability, linear systems, observability, robust control, uncertain systems.**

I. Introduction

In this paper, we examine the control problem for uncertain systems because almost every system contains some uncertainty. It is useful to classify the existing results on the robust control problem of uncertain systems into two categories. The first category includes several results that provide conditions depending on the bounds of uncertain parameters. On the other hand, the second category includes results that provide conditions that are independent of the bounds of uncertain parameters but dependent on their locations in system matrices. In this paper, we specifically address the second category.

Stabilizability is a particularly important concept in control theory. Stabilizability conditions in the first category are often used to determine the permissible bounds of uncertain parameters for the stabilization of uncertain systems. When uncertain parameters exceed particular values, guidelines for designing the controller are usually lacking in the first category. On the other hand, stabilizability conditions in the second category [1] can be verified simply by examining the locations of uncertain parameters in given system matrices. Once a system satisfies the stabilizability conditions, a stabilizing controller can be constructed, however large the given bounds of uncertain parameters might be. Moreover, we can redesign the controller to improve robustness simply by modifying the design parameter when uncertain parameters exceed the particular bounds given beforehand.

Focusing on the second category, Wei introduced a particular geometric configuration called an antisymmetric stepwise configuration (ASC) [2] or a generalized antisymmetric stepwise configuration (GASC) [3], and proved that a linear time-varying or time-invariant uncertain system is stabilizable via linear control, independently of the given bounds of uncertain parameters, if and only if the system has an ASC or a GASC, respectively.

Controllability is also an important concept in control theory [4]. Recently, the problem of the robust controllability of uncertain systems has been studied [5]–[17]. The system under consideration in [5]–[8] is of the form with system matrices having sign-invariant intervals, which is called a linear sign-invariant interval system. For such a system, attention has been focused on the cases of a single-input system [5], a descriptor system [6], [7], and a multi-input system [8]. For time-varying uncertain systems with bounded uncertain parameters, robust controllability was investigated in [9]–[17]. However, all the aforementioned results [5]–[17] belong to the first category. This means that all the conditions presented in those papers for robust controllability depend on the bounds of uncertain parameters.

On the other hand, there are a few reports [18], [19] on the robust controllability problem in the second category. The concept of controllability invariance has been introduced and studied in [18], [19], which means that a linear uncertain system is controllable regardless of the bounds of uncertain parameters. The condition for controllability invariance of a time-varying uncertain system was derived in [18]. However, the robust controllability condition in [18] is much stricter than the condition of the GASC with respect to the permissible locations of uncertain parameters in system matrices. For a time-invariant uncertain system, it was shown in [19] that the system is controllability-invariant if and only if the system has a GASC. Consequently, we found that robust stabilizability is equivalent to controllability invariance for a certain class of linear time-invariant uncertain systems. Note that the definition of controllability invariance [19] is different from that of structural controllability [20]–[23].

It is usual that the state variables of systems are measured through outputs, hence, only limited parts of the state variables

Manuscript received February 17, 2010; revised July 15, 2010 and October 28, 2010; accepted October 28, 2010. Date of publication December 03, 2010; date of current version August 03, 2011. This work was supported in part by the Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for Young Scientists (22760318) and in part by the Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientists (20860040). Recommended by Associate Editor D. Arzelier.

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Digital Object Identifier 10.1109/TAC.2010.2096013

can be used directly. It is well known that observability plays a crucial role in the stabilization of systems with limited measurable state variables [24]. Recently, the notation for the observability of a class of uncertain systems has been introduced and studied in [25]–[27]. The uncertainties considered in [25], [26] and [27] are time-varying parameters and stochastic parameters, respectively. Since these parameters are assumed to be bounded, the conditions obtained in [25]–[27] depend on the bounds of uncertain parameters and belong to the first category. To the best of the authors' knowledge, the condition for robust observability in the second category has not yet been derived. Therefore, in this paper, we determine the permissible locations of uncertain parameters in system matrices for a linear uncertain system to be controllable and observable independently of the bounds of uncertain parameters, i.e., so-called controllability and observability invariance.

The objective of this paper is to derive a necessary and sufficient condition of controllability and observability invariance for a class of linear time-invariant uncertain systems. For this purpose, we first consider the conditions that must be satisfied for a controllability-invariant system to also be observability-invariant. It is shown here that a system with a GASC must have at least two outputs to be observability-invariant. For this reason, we focus on a class of systems consisting of one input and two outputs. Then, we propose a novel configuration called a complete generalized antisymmetric stepwise configuration (CGASC), which includes the GASC. The CGASC shown here consists of one input and two outputs, while a GASC consists of only one input. It is shown here that a linear uncertain system is controllability- and observability-invariant if and only if the system has a CGASC.

Furthermore, we discuss the controllability and observability of systems consisting of two inputs and one output. We also introduce a dual complete generalized antisymmetric stepwise configuration (DCGASC), and show that a system with two inputs and one output is controllability- and observability-invariant if and only if the system has a DCGASC. The results obtained here show that the configuration of permissible locations of uncertain parameters has an interesting antisymmetric form between the CGASC and the DCGASC.

This paper is organized as follows. Some notation and system definitions are given in Section II. The fundamental lemmas leading to the main result are provided in Section III. The main results are given in Section IV. An illustrative example is presented in Section V. Some concluding remarks are given in Section VI. The proofs of some results are given in the appendices.

II. System Definitions

We consider a linear time-invariant uncertain system that is a set of systems described by the state equation

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^s$ is the output. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{s\times n}$ are system coefficient matrices, each entry of which belongs to either $\mathbb{R}$, $\mathbb{R}\backslash\{0\}$, or $\{0\}$. Throughout this paper, the notations $*$, θ, and 0 always denote entries belonging to $\mathbb{R}$, $\mathbb{R}\backslash\{0\}$, and $\{0\}$, respectively. Note that θ and 0 always denote a nonzero entry and the zero entry, respectively. Each entry of $*$ and θ is uncertain and may take an arbitrary value in $\mathbb{R}$ and $\mathbb{R}\backslash\{0\}$, respectively. Each entry in the system coefficient matrices is assumed to be known to be either $*$, θ, or 0. In other words, we assume that each entry of the system matrices is known to belong to either $\mathbb{R}$, $\mathbb{R}\backslash\{0\}$, or $\{0\}$. Let the notations $a_{i,j} \equiv 0$ and $a_{i,j} \not\equiv 0$ be always used if $a_{i,j}$ belongs to $\{0\}$ and $\mathbb{R}\backslash\{0\}$, respectively. Note that we can classify a set of systems by examining the locations of the entries. For example, we consider the following sets of systems: Let Σ_{A_1} be a set of systems such that $a_{2,1} \equiv 0$, $a_{3,1} \equiv 0$, and $a_{3,2} \equiv 0$. Let Σ_{A_2} be a set of systems such that $a_{3,1} \equiv 0$ and $a_{3,2} \equiv 0$. Let Σ_{A_3} be a set of systems such that $a_{2,1} \equiv 0$ and $a_{3,2} \equiv 0$

$$A_1 = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}, \; A_2 = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}, \; A_3 = \begin{bmatrix} * & * & * \\ 0 & * & * \\ * & 0 & * \end{bmatrix}.$$

Note that $\Sigma_{A_1} \subset \Sigma_{A_2}$ and $\Sigma_{A_1} \subset \Sigma_{A_3}$, but $\Sigma_{A_2} \not\subset \Sigma_{A_3}$ and $\Sigma_{A_3} \not\subset \Sigma_{A_2}$.

Let the set of systems whose every entry is $*$ be denoted by Σ. The transpose and determinant of $A \in \mathbb{R}^{n\times n}$ are denoted by A' and $\det(A)$, respectively.

Now, we introduce the concept of controllability invariance, defined in [19].

Definition 1: A set of systems is said to be controllability-invariant if all the systems in the set are controllable. This means that a set of systems is controllability-invariant if all the systems in the set satisfy

$$rank\,[\,(A-\lambda I) \quad B\,] = n, \quad \text{for } \forall\lambda \in \mathbb{C}. \tag{2}$$

It is seen from Definition 1 that a linear uncertain system is controllability-invariant if the system is controllable for any fixed values of uncertain parameters. Note that the controllability invariance is independent of the bounds of uncertain parameters.

It is obvious that a system must have at least one control input for its controllability. We first assume that $B \in \mathbb{R}^{n\times 1}$. It was shown in [3] that a necessary condition for a set of systems to satisfy condition (2) is that there should be nonzero entries in all the n rows and in at least n columns of the matrix $M \in \mathbb{R}^{n\times(n+1)}$ defined as

$$M = [\,A \quad B\,]. \tag{3}$$

In accordance with this fact, we focus on a certain class of systems called a standard system, defined in [3].

Definition 2: A set of systems $\subset M \in \mathbb{R}^{n\times(n+1)}$ is called a standard system if $M = \{m_{ij}\}$ has the property such that $m_{ii+1} = \theta$ for each $i = 1, 2, \ldots, n$. The standard system $\subset M$ is denoted by Σ_M.

Note that a standard system Σ_M has no restriction on nonsuperdiagonal entries of M

$$\Sigma_M = \begin{bmatrix} * & \theta & * & * & * & * \\ * & * & \theta & * & * & * \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ * & * & * & * & \theta & * \\ * & * & * & * & * & \theta \end{bmatrix}. \tag{4}$$

Next, we introduce a generalized antisymmetric stepwise configuration, defined in [3].

Definition 3: A standard system Σ_M is said to have a generalized antisymmetric stepwise configuration (GASC) if the following two conditions hold:

(i) If $h \geq g+2$ and $m_{gh} \not\equiv 0$, then $m_{uv} \equiv 0$ for all $u \geq v$, $u \leq h-1$, and $v \leq g$.

(ii) $\det(M^r) = m_{12}m_{23}\cdots m_{nn+1}$, where M^r is the right submatrix of M defined by

$$M^r = \begin{bmatrix} m_{12} & m_{13} & \cdots & m_{1n+1} \\ m_{22} & m_{23} & \cdots & m_{2n+1} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n2} & m_{n3} & \cdots & m_{nn+1} \end{bmatrix}. \tag{5}$$

If Σ_M has a GASC, then all the systems in Σ_M are also said to have a GASC. The set of all systems having a GASC is denoted by Σ_G. An example of Σ_G is described by (6). If there is an entry $m_{gh} \not\equiv 0$ in the upper part of the superdiagonal θ-entries ($m_{gh} \not\equiv 0$ and $h \geq g+2$), all the entries enclosed by the solid line are equal to zero by condition (i) of Definition 3 ($m_{uv} \equiv 0$ for all $u \geq v$, $u \leq h-1$, and $v \leq g$). Furthermore, condition (ii) of Definition 3 is used to determine the configurations of the uncertain entries surrounded by the dashed line.

$$M = \begin{bmatrix} 0 & \theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 0 & 0 & m_{gh} & 0 & 0 & 0 \\ 0 & 0 & * & \theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta & 0 & 0 & 0 \\ * & * & * & * & * & * & \theta & 0 & 0 \\ * & * & * & * & * & * & * & \theta & 0 \\ * & * & * & * & * & * & * & * & \theta \end{bmatrix} \tag{6}$$

Next, we define observability invariance as well as controllability invariance.

Definition 4: A set of systems is said to be observability-invariant if all the systems in the set are observable. This means that a set of systems is observability-invariant if all the systems in the set satisfy

$$rank \begin{bmatrix} C \\ (A-\lambda I) \end{bmatrix} = n, \quad \text{for } \forall \lambda \in \mathbb{C}. \tag{7}$$

It is seen that a linear uncertain system is observability-invariant if the system is observable for any fixed values of uncertain parameters. Note that the observability invariance is also independent of the bounds of uncertain parameters.

It is clear that a system must have at least one output for its observability. We firstly assume that $C \in \mathbb{R}^{1\times n}$. According to Definition 4, it is apparent that if a set of systems is observability-invariant, then there must exist nonzero entries in all the n columns and in at least n rows of the matrix $N \in \mathbb{R}^{(n+1)\times n}$ defined as

$$N = \begin{bmatrix} C \\ A \end{bmatrix}. \tag{8}$$

Hence, we also focus on the standard system defined below for $N \in \mathbb{R}^{(n+1)\times n}$.

Definition 5: A set of systems $\subset N \in \mathbb{R}^{(n+1)\times n}$ is called a standard system if $N = \{n_{ij}\}$ has the property such that $n_{ii} = \theta$ for each $i = 1, 2, \ldots, n$. The standard system $\subset N$ is denoted by Σ_N.

Note that a standard system Σ_N has no restriction on nondiagonal entries of N.

$$\Sigma_N = \begin{bmatrix} \theta & * & * & * & * \\ * & \theta & * & * & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ * & * & * & \theta & * \\ * & * & * & * & \theta \\ * & * & * & * & * \end{bmatrix}. \tag{9}$$

III. Fundamental Lemmas

In this section, we provide the preliminary lemmas used in deriving the main results. A necessary and sufficient condition for a standard system Σ_M to be controllability-invariant was proved in [19].

Lemma 1 ([19]): A standard system Σ_M is controllability-invariant if and only if Σ_M has a GASC, where a GASC is given by Definition 3.

Next, we consider the conditions that must be satisfied for a standard system Σ_N to be observability-invariant. In the following, we show the necessary conditions under which a standard system Σ_N is observability-invariant.

Lemma 2: If a standard system Σ_N is observability-invariant, then all the following conditions must hold:

(i) If $h \geq g+1$ and $n_{gh} \not\equiv 0$, then $n_{hg} \equiv 0$.

(ii) If $h \geq g+1$ and $n_{gh} \not\equiv 0$, then $n_{uv} \equiv 0$ for all $u \geq v-1$, $u \geq h+1$, and $v \geq g$.

(iii) $\det(N^u) = n_{11}n_{22}\cdots n_{nn}$, where N^u is the upper submatrix of N defined as

$$N^u = \begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1n} \\ n_{21} & n_{22} & \cdots & n_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ n_{n1} & n_{n2} & \cdots & n_{nn} \end{bmatrix}. \tag{10}$$

Proof: The proof is given in Appendix A. ∎

As is shown in (11), if there is an entry $n_{gh} \not\equiv 0$ in the upper part of the diagonal θ-entries ($n_{gh} \not\equiv 0$ and $h \geq g+1$), then the entry enclosed by the solid line is equal to zero by condition (i) of Lemma 2 ($n_{hg} \equiv 0$); moreover, by condition (ii) of Lemma

2 ($n_{uv} \equiv 0$ for all $u \geq v-1$, $u \geq h+1$, and $v \geq g$), all the entries surrounded by the dashed line are equal to zero

$$N = \begin{bmatrix} \theta & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \theta & 0 & 0 & n_{gh} & 0 & 0 \\ * & * & \theta & 0 & 0 & 0 & 0 \\ * & * & * & \theta & 0 & 0 & 0 \\ * & \boxed{0} & * & * & \theta & 0 & 0 \\ * & 0 & 0 & 0 & 0 & \theta & 0 \\ * & 0 & 0 & 0 & 0 & 0 & \theta \\ * & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (11)$$

It is apparent from (6) that, for a system in Σ_G, the entries enclosed by the dashed line in (11) are not necessarily equal to zero. It is important to note that condition (ii) of Lemma 2 is additional to the conditions of Definition 3. Comparing (6) with (11), we see that a system having a GASC cannot be observability-invariant under the assumption of one output ($C \in \mathbb{R}^{1\times n}$). This property makes the observability invariance problem nontrivial.

Next, we consider how the output coefficient matrix C must be constructed for Σ_G to be observability-invariant. We then give the necessary conditions under which Σ_G is observability-invariant.

Lemma 3: Let $\underline{g}$ denote the lowest row that contains a nonzero uncertain entry in the upper part of the θ-entries. If Σ_G is observability-invariant, then all the following conditions hold.

(i) If $1 \leq \underline{g} \leq n-2$, then C must contain both the row vectors given by

$$[\theta_1 \quad * \quad * \quad \cdots \quad \cdots \quad * \quad *] \quad (12)$$

$$[0 \quad \cdots \quad 0 \quad \theta_{\underline{g}+1} \quad 0 \quad \cdots \quad 0]. \quad (13)$$

(ii) If $\underline{g} = n-1$, then C must contain the row vector given by (12).

(iii) If $\underline{g} = 0$, then C must contain the row vector given by (14).

$$[\theta_1 \quad 0 \quad 0 \quad \cdots \quad \cdots \quad 0 \quad 0]. \quad (14)$$

Proof: The proof is given in Appendix B. ■

In the case of $1 \leq \underline{g} \leq n-2$, at least two outputs $C \in \mathbb{R}^{2\times n}$ are necessary for Σ_G to be observability-invariant. For this reason, we assume $C \in \mathbb{R}^{2\times n}$ in the subsequent discussion. We thus write C as

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} := \begin{bmatrix} * & \cdots & * \\ * & \cdots & * \end{bmatrix} \in \mathbb{R}^{2\times n}. \quad (15)$$

IV. Main Results

It is seen from Lemma 2 that all the permissible locations of $*$ for a set of systems to be controllability-invariant are not necessarily allowable for the set of systems with one output to be observability-invariant. That means the observability invariance of Σ_G might be lost under the assumption of Σ_G with only one output. It is also seen from Lemma 3 that Σ_G must contain at least two outputs to be observability-invariant. The objective of this paper is to reveal a synthesized configuration that includes a GASC for controllability and observability invariance. Toward this objective, the system coefficient matrices are assumed here to be $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and $C \in \mathbb{R}^{2\times n}$, because Σ_G must have at least one input and two outputs to be controllability- and observability-invariant. Then, we introduce below a standard system for a set of systems with one input and two outputs.

Definition 6: Let $P \in \mathbb{R}^{(n+1)\times(n+1)}$ and $\hat{P} \in \mathbb{R}^{(n+2)\times(n+1)}$ be defined as

$$P = \begin{bmatrix} C_1 & 0 \\ A & B \end{bmatrix} \quad \text{and} \quad \hat{P} = \begin{bmatrix} C_1 & 0 \\ A & B \\ C_2 & 0 \end{bmatrix} \quad (16)$$

respectively. A set of systems $\subset \hat{P}$ is called a standard system if $P = \{p_{ij}\}$ has the property such that $p_{ii} = \theta$ for each $i = 1, 2, \ldots, n+1$. The standard system $\subset \hat{P}$ is denoted by Σ_P.

Note that a standard system Σ_P has no restrictions except on the diagonal entries of P

$$P = \begin{bmatrix} \theta & * & * & 0 \\ * & \ddots & * & * \\ * & * & \ddots & * \\ * & * & * & \theta \end{bmatrix} \quad (17)$$

$$C_2 = [* \quad \cdots \quad \cdots \quad *]. \quad (18)$$

Next, we introduce a key configuration to lead to the main result, i.e., Theorem 1.

Definition 7: A standard system Σ_P is said to have a complete generalized antisymmetric stepwise configuration (CGASC) if all the following conditions hold.

(i) If $h \geq g+1$ and $p_{gh} \not\equiv 0$, then $p_{hg} \equiv 0$ and $p_{uv} \equiv 0$ for all $u \geq v-1$, $u \leq h$, and $v \leq g-1$.

(ii) $\det(P) = p_{11}p_{22}\cdots p_{n+1n+1}$.

(iii) For $\underline{g} := \min\{g | p_{gh} \not\equiv 0 (h \geq g+1)\}$, if $1 \leq \underline{g} \leq n-1$, then

$$C_2 = [0 \quad \cdots \quad 0 \quad \theta_{\underline{g}} \quad 0 \quad \cdots \quad 0].$$

An example of a CGASC is shown below. If there is an entry $p_{gh} \not\equiv 0$ in the upper part of the diagonal θ-entries ($p_{\underline{g}h} \not\equiv 0$ and $h \geq g+1$), all the entries enclosed by the solid line are equal to zero by condition (i) of Definition 7 ($p_{hg} \equiv 0$ and $p_{uv} \equiv 0$ for all $u \geq v-1$, $u \leq h$, and $v \leq g-1$). In addition, condition (ii) of Definition 7 determines the configurations of the uncertain entries surrounded by the dashed line. Furthermore, by condition (iii) of Definition 7, the location of the nonzero entry θ in the row vector C_2 is fixed

$$P = \begin{bmatrix} \theta & * & * & * & * & * & 0 & 0 \\ 0 & \theta & * & * & * & * & 0 & 0 \\ 0 & 0 & \theta & * & * & p_{\underline{g}h} & 0 & 0 \\ 0 & 0 & 0 & \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & \theta & 0 & 0 \\ * & * & * & * & * & * & \theta & 0 \\ * & * & * & * & * & * & * & \theta \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \underline{g} \\ \\ \\ \\ \\ \\ \end{matrix} \quad (19)$$

$$C_2 = [\,0 \quad 0 \quad \theta_{\underline{g}} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,]$$

Now, we state the main result.

Theorem 1: A standard system Σ_P is controllability- and observability-invariant if and only if Σ_P has a CGASC, where a CGASC is given by Definition 7.

Proof: Note that p_{gh}, p_{1h}, and p_{gn+1} correspond to a_{g-1h}, c_h^1, and b_{g-1}, respectively. Neglecting the condition on C and noting the relation between A and B, we see that the conditions of the CGASC are equivalent to those of the GASC except for $p_{hg} \equiv 0$ in condition (i) of Definition 7. In fact, it was shown in [19] that $p_{hg} \equiv 0$ is also a necessary condition, and that this condition can be included in condition (ii) of Definition 3. Therefore, it is apparent from Lemma 1 that Σ_P is controllability-invariant if and only if Σ_P has a CGASC.

On the other hand, noting the relation between A and C, we see from Lemma 2 that conditions (i) and (ii) in Theorem 1 are necessary for Σ_P to be observability-invariant. Furthermore, because Σ_P must have a GASC to ensure controllability invariance, it follows from Lemma 3 that condition (iii) of Theorem 1 is necessary for Σ_P to be observability-invariant. Consequently, it is apparent that if Σ_P is observability-invariant, then all the conditions in Theorem 1 must be satisfied.

To complete the proof of Theorem 1, it remains to be shown that if all the conditions in Theorem 1 are satisfied, then Σ_P is observability-invariant. The sufficiency of Theorem 1 for observability invariance is given in Appendix C. ■

So far, we have focused on a class of systems with one input and two outputs. Noting that there is an imbalance between the dimensions of the input and output, we also examine here the controllability and observability for a class of systems with two inputs and one output. In the subsequent discussion, we assume $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 2}$, and $C \in \mathbb{R}^{1\times n}$.

Let $B \in \mathbb{R}^{n\times 2}$ be given by

$$B = [\,B_1 \quad B_2\,] := \begin{bmatrix} * & * \\ \vdots & \vdots \\ * & * \end{bmatrix} \in \mathbb{R}^{n\times 2}. \tag{20}$$

Definition 8: Let $Q \in \mathbb{R}^{(n+1)\times(n+1)}$ and $\hat{Q} \in \mathbb{R}^{(n+1)\times(n+2)}$ be defined by

$$Q = \begin{bmatrix} B_1 & A \\ 0 & C \end{bmatrix} \quad \text{and} \quad \hat{Q} = \begin{bmatrix} B_1 & A & B_2 \\ 0 & C & 0 \end{bmatrix} \tag{21}$$

respectively. A set of systems $\subset \hat{Q}$ is called a standard system if $Q = \{q_{ij}\}$ has the property such that $q_{ii} = \theta$ for each $i = 1, 2, \ldots, n+1$. The standard system $\subset \hat{Q}$ is denoted by Σ_Q.

Note that a standard system Σ_Q has no restrictions except on the diagonal entries of Q. We next introduce the dual structure of a CGASC for Σ_Q.

Definition 9: A standard system Σ_Q is said to have a dual complete generalized antisymmetric stepwise configuration (DCGASC) if all the following conditions hold:

(i) If $h \le g+1$ and $q_{gh} \not\equiv 0$, then $q_{hg} \equiv 0$ and $q_{uv} \equiv 0$ for all $u \le v+1$, $u \le h-1$, and $v \le g$.

(ii) $\det(Q) = q_{11}q_{22}\cdots q_{n+1n+1}$.

(iii) For $\bar{h} := \max\{h | q_{gh} \not\equiv 0 (g > h)\}$, if $1 \le \bar{h} \le n-1$, then

$$B_2 = [0 \quad \cdots \quad 0 \quad \theta_{\bar{h}} \quad 0 \quad \cdots \quad 0]'.$$

If there is an entry $q_{g\bar{h}} \not\equiv 0$ in the lower part of the diagonal θ-entries ($h \le g+1$ and $q_{g\bar{h}} \not\equiv 0$), all the entries enclosed by the solid line are equal to zero by condition (i) of Definition 9 ($q_{hg} \equiv 0$ and $q_{uv} \equiv 0$ for all $u \le v+1$, $u \le h-1$, and $v \le g$). In addition, condition (ii) of Definition 9 determines the configurations of the uncertain entries surrounded by the dashed line. Furthermore, by condition (iii) of Definition 9, the location of the nonzero entry θ in the vector B_2 is fixed

$$Q = \begin{bmatrix} \theta & 0 & 0 & 0 & 0 & 0 & * & * \\ * & \theta & 0 & 0 & 0 & 0 & * & * \\ * & * & \theta & 0 & 0 & 0 & * & * \\ * & * & * & \theta & * & * & * & * \\ * & * & * & 0 & \theta & * & * & * \\ * & * & q_{g\bar{h}} & 0 & 0 & \theta & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta \end{bmatrix} B_2 = \begin{bmatrix} 0 \\ 0 \\ \theta_{\bar{h}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

Corollary 1: A standard system Σ_Q is controllability- and observability-invariant if and only if Σ_Q has a DCGASC, where a DCGASC is given by Definition 9.

Proof: Consider the following set of systems:

$$\dot{x} = A'x + C'u, \qquad y = B'x. \tag{23}$$

We denote the above set of systems by Σ'. Considering

$$rank\,[\,A' - \lambda I \quad C'\,] = rank \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \tag{24}$$

we see that Σ' is controllability-invariant if and only if Σ is observability-invariant. Likewise, we see from (25) that Σ' is observability-invariant if and only if Σ is controllability-invariant.

$$rank \begin{bmatrix} A' - \lambda I \\ B' \end{bmatrix} = rank\,[\,A - \lambda I \quad B\,]. \tag{25}$$

Note that Σ_Q has a DCGASC if and only if Σ'_Q has a CGASC. Therefore, it follows from Theorem 1 that Σ_Q is controllability- and observability-invariant if and only if the system has a DCGASC. ■

Comparing (19) with (22), we see that there is an interesting antisymmetric form between the CGASC and the DCGASC in terms of the permissible locations of uncertain parameters. For time-varying systems, this property might be lost.

V. Illustrative Example

Designing a network structure of information flows arises in many systems including communication systems, formation moving (flying) systems, molecular biological systems, and genetic systems. Consider the two network structures shown in Figs. 1 and 2.

For simplicity, we assume that each subject X_i is governed by the state equation $\dot{x}_i(t) = v_i(t)$ for $i = 1, \ldots, 4$, where $x_i(t) \in \mathbb{R}$ is the state, $v_i(t) \in \mathbb{R}$ is the input of the information flow, and the output is given by $y_i(t) = x_i(t)$. Let $X_i \longrightarrow X_j$ denote the fact that the output y_i of X_i is employed in the input v_j of X_j. Let $X_i \cdots\succ X_j$ denote the fact that $\varepsilon_{ji}y_i(t)$ is employed in the input v_j of X_j, where $\varepsilon_{ji} \not\equiv 0$ is an uncertain parameter. Moreover, we assume that X_5 is a controllable object that is governed by the state equation $\dot{x}_5(t) = u(t)$, where $x_5(t) \in \mathbb{R}$ is the state and $u(t) \in \mathbb{R}$ is the control variable constructed from the information input of X_5.

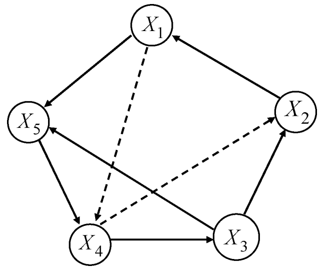

Fig. 1. Network 1.

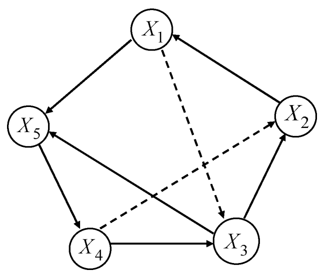

Fig. 2. Network 2.

Network 1 in Fig. 1 can be described by the following state equation:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon_{24} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \varepsilon_{41} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x. \tag{26}$$

In contrast, Network 2 in Fig. 2 is described by the following state equation:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon_{24} & 0 \\ \varepsilon_{31} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x. \tag{27}$$

We see from (26) and (27) that Network 1 has a CGASC, while Network 2 has no CGASC. Consequently, we see from Theorem 1 that Network 1 is controllable and observable for any fixed values of uncertain parameters. Moreover, Network 2 might lose its controllability or observability for certain values of uncertain parameters. In fact, if we choose $\lambda = -1$, $\varepsilon_{24} = 1$, and $\varepsilon_{31} = -1$, then we have $rank[(A - \lambda I) \quad B] = 4$. This means that Network 2 is uncontrollable.

VI. Conclusion

In this research, we studied the controllability and observability of linear time-invariant uncertain systems. In particular, we examined the conditions for controllability and observability invariance, which means that a linear uncertain system is controllable and observable in the usual sense for any fixed values of uncertain parameters. The obtained conditions for controllability and observability invariance are independent of the bounds of uncertain parameters, but dependent on the locations of uncertain parameters. The controllability and observability of the considered system are determined by the localization of uncertain parameters.

We first showed that to ensure the observability invariance for a system with a generalized antisymmetric stepwise configuration, the system must have at least two outputs. Then, we focused on a certain class of uncertain systems called a standard system consisting of one input and two outputs. Next, we introduced a complete generalized antisymmetric stepwise configuration (CGASC). We showed that a standard system is controllability- and observability-invariant if and only if the system has a CGASC. Furthermore, we introduced the dual structure of a CGASC for systems consisting of two inputs and one output, and derived a necessary and sufficient condition for a standard system to be controllability- and observability-invariant. We found that there is an interesting antisymmetric form between the CGASC and the DCGASC concerning the permissible locations of uncertain parameters.

For a standard system, the conditions derived here classify uncertain systems into two groups. One consists of uncertain systems whose controllability and observability depend on the bounds of uncertain parameters. The other consists of uncertain systems whose controllability and observability are independent of the bounds of the uncertain parameters. From a basic point of view, we focused on the simplest system configuration such as one input and two outputs or two inputs and one output for controllability and observability invariance. Using the obtained criteria, we found that if a standard system has no CGASC, then more inputs or outputs are necessary for the system to be controllability- and observability-invariant. It is beyond the scope of this paper to consider how we should construct additional inputs or outputs for controllability and observability invariance. This is a problem that can be investigated in the future.

Furthermore, it is worth noting that the problem addressed here is strongly connected to the problem of designing a stabilizing controller for uncertain systems with limited measurable state variables. Even if an uncertain system has uncontrollable stable eigenvalues, they might become unstable owing to the variations in uncertain parameters. Then, we find that if an uncertain system is stabilizable using an output feedback controller, independently of the bounds of uncertain parameters, then the system must be controllability- and observability-invariant. Thus, the conditions derived here are necessary to develop a controller/observer synthesis method for uncertain systems in the second category. The output feedback stabilization problem of uncertain systems, irrespective of the bounds of uncertain parameters, is a problem that we plan to consider next.

Appendix A
Proof of Lemma 2

First, we show the necessity of conditions (i) and (ii) in Lemma 2. Suppose that a standard system Σ_N is observability-invariant. Then the following system (28) is also observability-invariant, where $g < h$, $u > v$, and $v \geq g$.

$$N = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & n_{gh} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdot & \ddots & \ddots & 0 \\ \vdots & n_{uv} & \cdot & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}. \tag{28}$$

Then, we assume that the rank condition (7) in Definition 4 holds for the case where N contains only two uncertain entries n_{gh} and n_{uv}. Let $\bar{N}$ be defined by

$$\bar{N} = \begin{bmatrix} C \\ A - \lambda I \end{bmatrix}. \tag{29}$$

Let $\bar{N}_k$ denote the $k \times k$ upper left submatrix of $\bar{N}$. For system (28), it follows that $rank(A - \lambda I) = n - 1$ for λ by taking the eigenvalues of A. For this reason, if system (28) is observability-invariant, then $\bar{N}_n$ must be of full rank, that is, $\det(\bar{N}_n) \neq 0$ must hold.

Now, we consider each case separately, depending on the location of n_{gh}: [I] $g = 1$ and [II] $g \neq 1$.

[I]: Consider the case of $g = 1$. Note that the characteristic polynomial of A for system (28) is described by

$$(-\lambda)^{n-u+v-1}\left\{(-\lambda)^{u-v+1} + n_{uv}\right\} = 0. \tag{30}$$

In the following, we derive the conditions that ensure $\det(\bar{N}_n) \neq 0$ for all the roots of (30). We further separate this case according to the location of n_{uv}: (i) $u \leq h$ and (ii) $u > h$.

(i) When $u \leq h$, it follows that $\det(\bar{N}_n) = \det(\bar{N}_h)$ and $\bar{N}_h$ is described by

$$\bar{N}_h = \begin{bmatrix} 1 & 0 & \cdot & 0 & n_{1h} \\ -\lambda & 1 & \cdot & \cdot & 0 \\ \cdot & \ddots & \ddots & \cdot & \cdot \\ \cdot & n_{uv} & \ddots & \ddots & 0 \\ 0 & \cdot & \cdot & -\lambda & 1 \end{bmatrix}. \tag{31}$$

Then, $\det(\bar{N}_h)$ is obtained as follows:

(a) For $u \neq h, v \neq 1$,

$$\det(\bar{N}_h) = 1 + (-1)^{h+1} n_{1h}(-\lambda)^{h-u+v-2} \times \left\{(-\lambda)^{u-v+1} + n_{uv}\right\}. \tag{32}$$

(b) For $u = h, v = 1$,

$$\det(\bar{N}_h) = 1 + (-1)^{h+1} n_{1h}\left\{(-\lambda)^{h-1} + n_{uv}\right\}. \tag{33}$$

By taking $\lambda = \sqrt[u-v+1]{(-1)^{u-v+2} n_{uv}}$ in (30), the second term on the right-hand side of both (32) and (33) vanishes. Hence, it follows that $\det(\bar{N}_n) = 1 \neq 0$ in cases (a) and (b).

On the other hand, taking $\lambda = 0$ in (30) yields $\det(\bar{N}_n) = 1$ in (32). Substituting $\lambda = 0$ into (33) yields

$$\det(\bar{N}_n) = 1 + (-1)^{h+1} n_{1h} n_{uv}. \tag{34}$$

Therefore, if $\det(\bar{N}_n) = 1$ for $n_{1h} \not\equiv 0$, then $n_{uv} \equiv 0$ $(u = h, v = g = 1)$ must hold from (34).

(ii) When $u > h$, $\bar{N}_n$ is described by

$$\bar{N}_n = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdot & n_{1h} & 0 & \cdot & \cdot & 0 \\ -\lambda & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \ddots & \ddots & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & -\lambda & 1 & 0 & \cdot & \cdot & 0 \\ \hline 0 & \cdot & \cdot & \cdot & 1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & -\lambda & 1 & \cdot & \cdot \\ \cdot & n_{uv} & \cdot & \cdot & \cdot & \ddots & \ddots & 0 \\ 0 & \cdot & \cdot & 0 & 0 & \cdot & -\lambda & 1 \end{array}\right]. \tag{35}$$

Then, we have

$$\det(\bar{N}_n) = \bar{N}_h = 1 + (-1)^{h+1} n_{1h}(-\lambda)^{h-1}. \tag{36}$$

If $\det(\bar{N}_n) = 1$ for $n_{1h} \not\equiv 0$, then we must have $\lambda = 0$. Therefore, $n_{uv} \equiv 0$ $(u > h)$ must hold so that all the roots of (30) are equal to zero.

[II]: Consider the case of $g \neq 1$. Let $\bar{N}_n$ be decomposed into the block matrices shown in (37). Let $\bar{N}_c$ denote the $(2, 2)$ block matrix in (37). We separate cases according to the location of n_{uv} in the same way as for [I], see (37) at the top of the next page.

i) When $u \leq h$, n_{uv} is located in the $(2, 2)$ block matrix. Note that $\bar{N}_c$ has the same structure as (31). Likewise, we see that if $\det(\bar{N}_n) = \det(\bar{N}_c) = 1 \neq 0$, then $n_{uv} \equiv 0$ $(u = h, v = g)$ must hold.

ii) When $u > h$, we have

$$\begin{aligned} \det(\bar{N}_n) &= \det(\bar{N}_h) = \det(\bar{N}_c) \\ &= 1 + (-1)^{h-g+1} n_{gh}(-\lambda)^{h-g-1}. \end{aligned} \tag{38}$$

It is clear that $\lambda = 0$ must hold so that we obtain $\det(\bar{N}_n) = 1 \neq 0$ for $n_{gh} \not\equiv 0$. In this case, the characteristic polynomial of A is described by

$$(-\lambda)^{v-1}\left\{(-\lambda)^{n-v+1} + (-1)^{u+v} n_{uv}(-\lambda)^{n-u}\right\} = 0. \tag{39}$$

Therefore, if all the roots of (39) are equal to zero, then $n_{uv} \equiv 0$ $(u > h)$ must hold.

From [I]-(i) and [II]-(i), the proof of condition (i) is complete. From [I]-(ii) and [II]-(ii), the proof of condition (ii) is also complete.

[III]: Finally, we show the necessity of condition (iii) in Lemma 2. Suppose that a standard system Σ_N is observability-invariant, then the rank condition (7) in Definition 4 must hold for every λ and $*$. For this reason, we must have $rankN^u = n$ for the case where λ and all the entries in the lowest row of A are equal to zero. Recall that N^u is defined as

$$N^u = \begin{bmatrix} \theta_1 & * & \cdots & * \\ * & \theta_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ * & \cdots & * & \theta_n \end{bmatrix}. \tag{40}$$

$$\bar{N}_n = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & \cdot & 0 & \cdot & 0 & 0 & \cdot & 0 \\ -\lambda & \ddots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \ddots & \ddots & 0 & \cdot & 0 & 0 & \cdot & 0 \\ \hline 0 & \cdot & 0 & 1 & \cdot & n_{gh} & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & -\lambda & \ddots & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & n_{uv(u\le h)} & \ddots & 1 & 0 & \cdot & 0 \\ \hline 0 & \cdot & 0 & 0 & \cdot & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & n_{uv(u>h)} & \cdot & \cdot & \ddots & \ddots & \cdot \\ 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & -\lambda & 1 \end{array}\right] \tag{37}$$

Expanding N^u with respect to the first row of N^u yields

$$\det(N^u) = \theta_1\Delta(\theta_1) + *\Delta(*) + \cdots + *\Delta(*), \tag{41}$$

where $\Delta(*)$ denotes the cofactor of $*$. All the terms of the above equation except for the first term contain no θ-entries, hence these terms must vanish. Then, $\det(N^u) = \theta_1\Delta(\theta_1)$ must hold so that we obtain $\det(N^u) \neq 0$ for any values of $*$. Likewise, expanding $\Delta(\theta_1)$ with respect to the first row of $\Delta(\theta_1)$, we must have $\det(N^u) = \theta_1\theta_2\Delta(\theta_2)$. Repeating the same discussion for the remaining cofactors, we see that $\det(N^u) = \theta_1 \cdots \theta_n$ must hold for a standard system Σ_N to be observability-invariant.

From [I], [II], and [III], the proof of Lemma 2 is complete.

Appendix B
Proof of Lemma 3

[I]: Consider the case of $1 \le g \le n-2$. Let a_{gh} be the rightmost entry of the lowest row containing an uncertain entry in the upper part of the θ-entries. It is found from Definition 3 that a system in Σ_G permits the existence of uncertain entries at locations such as a_{g+1g+1} and a_{nn} provided that $a_{gh} \not\equiv 0$ $(h \ge g+2)$ is given. Suppose that Σ_G is observability-invariant. Then the following system (42) in Σ_G must be also observability-invariant.

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & a_{gh} & \cdot \\ \cdot & \cdot & a_{g+1g+1} & \ddots & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 & \ddots & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & a_{nn} \end{bmatrix}. \tag{42}$$

In the following, we derive necessary conditions under which system (42) is observability-invariant. We adopt here the following proposition, which is well known as the Popov-Belevitch-Hautus theorem [28].

Proposition 1 ([28]): The following conditions are equivalent.

(i) $\forall\lambda \in \mathbb{C}, rank \begin{bmatrix} C \\ A-\lambda I \end{bmatrix} = n.$

(ii) $\forall\lambda \in \mathbb{C}, Ay = \lambda y, Cy = 0 \Rightarrow y = 0.$

Suppose that system (42) satisfies condition (ii) of Proposition 1. For simplicity, a_{g+1g+1} is subsequently denoted by a_{g+1}. From $Ay = \lambda y$, we have

$$y_2 = \lambda y_1, \tag{43}$$
$$y_3 = \lambda y_2, \tag{44}$$
$$\vdots$$
$$y_g = \lambda y_{g-1}, \tag{45}$$
$$y_{g+1} + a_{gh}y_h = \lambda y_g, \tag{46}$$
$$a_{g+1}y_{g+1} + y_{g+2} = \lambda y_{g+1}, \tag{47}$$
$$y_{g+3} = \lambda y_{g+2}, \tag{48}$$
$$\vdots$$
$$y_h = \lambda y_{h-1}, \tag{49}$$
$$\vdots$$
$$y_n = \lambda y_{n-1}, \tag{50}$$
$$a_{nn}y_n = \lambda y_n. \tag{51}$$

From (43)–(50), it follows that

$$y_2 = \lambda y_1 \tag{52}$$
$$y_3 = \lambda^2 y_1 \tag{53}$$
$$\vdots$$
$$y_g = \lambda^{g-1} y_1 \tag{54}$$
$$y_{g+1} = \lambda^g y_1 - a_{gh}\{\lambda^{h-g-2}(\lambda - a_{g+1})y_{g+1}\} \tag{55}$$
$$y_{g+2} = (\lambda - a_{g+1})y_{g+1} \tag{56}$$
$$y_{g+3} = \lambda(\lambda - a_{g+1})y_{g+1} \tag{57}$$
$$\vdots$$
$$y_h = \lambda^{h-g-2}(\lambda - a_{g+1})y_{g+1} \tag{58}$$
$$\vdots$$
$$y_n = \lambda^{n-g-2}(\lambda - a_{g+1})y_{g+1}. \tag{59}$$

(i) When $\lambda = 0$, we have $y_2 = \cdots = y_n = 0$ from (52)–(59). Then, we must have $y_1 = 0$ from $Cy = 0$ so that we obtain the unique solution $y = 0$. For this reason, C must have the row vector given by (12).

(ii) When $\lambda \neq 0$, it follows from (55) that

$$y_1 = \lambda^{-g}[1 + a_{gh}\lambda^{h-g-2}(\lambda - a_{g+1})]y_{g+1}. \tag{60}$$

Substituting (60) into (52)–(54), we obtain

$$\begin{aligned} y_2 &= \lambda^{1-g}[1 + a_{gh}\lambda^{h-g-2}(\lambda - a_{g+1})]y_{g+1} && (61)\\ y_3 &= \lambda^{2-g}[1 + a_{gh}\lambda^{h-g-2}(\lambda - a_{g+1})]y_{g+1} && (62)\\ &\vdots \\ y_g &= \lambda^{-1}[1 + a_{gh}\lambda^{h-g-2}(\lambda - a_{g+1})]y_{g+1}. && (63) \end{aligned}$$

We must have $y_{g+1} = 0$ from $Cy = 0$ provided that the unique solution $y = 0$ can be derived from (55)–(63). Hence, C must have the row vector given by (13). Therefore, it has been shown that if Σ_G is observability-invariant, then condition (i) of Lemma 3 must hold.

[II]: In the case of $\underline{g} = n-1$, all the entries in the lower part of the θ-entries are equal to zero from Definition 3. Suppose that condition (ii) of Proposition 1 holds for the following system:

$$A = \begin{bmatrix} 0 & \theta & * & * \\ \cdot & \ddots & \ddots & * \\ \cdot & \cdot & \ddots & \theta \\ 0 & \cdot & \cdot & 0 \end{bmatrix}. \tag{64}$$

From $Ay = \lambda y$, it follows that

$$\begin{aligned} y_2 &= \rho y_1 - (*y_3 + *y_4 + \cdots + *y_n) && (65)\\ y_3 &= \rho y_2 - (*y_4 + *y_5 + \cdots + *y_n) && (66)\\ &\vdots \\ y_{n-2} &= \rho y_{n-3} - (*y_{n-1} + *y_n) && (67)\\ y_{n-1} &= \rho y_{n-2} - *y_n && (68)\\ y_n &= \rho y_{n-1} && (69)\\ 0 &= \rho y_n && (70) \end{aligned}$$

where $\rho := \lambda/\theta$. When $\rho \neq 0$, we have $y_n = 0$ from (70). Then, from $\rho \neq 0$ and $y_n = 0$ in (69), it follows that $y_{n-1} = 0$. Repeating the same discussion for (65)–(68), we have $y_1 = \cdots = y_n = 0$. When $\rho = 0$, we have $y_2 = \cdots = y_n = 0$ from (65)–(69). Hence, C must have the row vector given by (12) so that we obtain $y_1 = 0$ from $Cy = 0$. Therefore, it has been shown that if system (64) is observability-invariant, then condition (ii) of Lemma 3 must hold.

[III]: In the case of $\underline{g} = 0$, all the entries in the upper part of the θ-entries are equal to zero from Definition 3. Suppose that condition (ii) of Proposition 1 holds for the following system:

$$A = \begin{bmatrix} * & \theta & 0 & 0 \\ \cdot & \ddots & \ddots & 0 \\ \cdot & \cdot & \ddots & \theta \\ * & \cdot & \cdot & * \end{bmatrix}. \tag{71}$$

From $Ay = \lambda y$, it follows that:

$$\begin{aligned} y_2 &= \rho y_1 - *y_1 && (72)\\ y_3 &= \rho y_2 - (*y_1 + *y_2) && (73)\\ &\vdots \\ y_{n-1} &= \rho y_{n-2} - (*y_1 + *y_2 + \cdots + *y_{n-2}) && (74)\\ y_n &= \rho y_{n-1} - (*y_1 + *y_2 + \cdots + *y_{n-1}). && (75) \end{aligned}$$

It is clear that all the terms on the right-hand side in (72)–(75) can be written in terms of ρ, $*$, and y_1 as follows:

$$\begin{aligned} y_2 &= f_2(\rho, *)y_1 && (76)\\ y_3 &= f_3(\rho, *)y_1 && (77)\\ &\vdots \\ y_{n-1} &= f_{n-1}(\rho, *)y_1 && (78)\\ y_n &= f_n(\rho, *)y_1 && (79) \end{aligned}$$

where $f_i(\rho, *)$ $(i = 2, \ldots, n)$ denote polynomials in terms of ρ and $*$. Hence, C must have the row vector given by (14) so that we obtain $y_1 = 0$ from $Cy = 0$. Therefore, it has been shown that if system (71) is observability-invariant, then condition (iii) of Lemma 3 must hold.

From [I], [II], and [III], the proof of Lemma 3 is complete.

Appendix C
Sufficiency of Theorem 1 for Observability-Invariance

Suppose that all the conditions in Theorem 1 are satisfied in the subsequent discussion. Let $D \in \mathbb{R}^{(n+2)\times n}$ be defined by

$$D = \begin{bmatrix} C_1 \\ A - \lambda I \\ C_2 \end{bmatrix}. \tag{80}$$

In the following, the proof is given separately for [I] $1 \leq \underline{g} \leq n-1$, [II] $\underline{g} = n$, and [III] $\underline{g} = 0$.

[I]: When $1 \leq \underline{g} \leq n-1$, Σ_P has a CGASC given by

$$\begin{bmatrix} \theta & * & \cdots & \cdots & & \cdots & * & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots & \vdots \\ 0 & \ddots & \theta & * & \cdots & \cdots & * & \vdots \\ \vdots & \ddots & 0 & \theta & * & \cdots & * & 0 \\ \vdots & \ddots & 0 & * & \theta & 0 & 0 & \vdots \\ \vdots & & \vdots & * & * & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * & * & \ddots & \ddots & 0 \\ * & \cdots & * & * & * & \ddots & \ddots & \theta \\ * & \cdots & * & * & * & \cdots & * & * \\ 0 & \cdots & 0 & \theta_{\underline{g}} & 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow \underline{g} \\ \\ \\ \\ \\ \\ \\ \end{matrix}. \tag{81}$$

Accordingly, D has the following structure:

$$\begin{array}{r} \\ \\ \\ \underline{g}\rightarrow \\ D= \end{array}\begin{bmatrix} \theta & * & \cdots & \cdots & & \cdots & * & 0 \\ -\lambda & \ddots & \ddots & \ddots & & & \vdots & \vdots \\ 0 & \ddots & \theta & * & \cdots & \cdots & * & \vdots \\ \vdots & \ddots & -\lambda & \theta & * & \cdots & * & 0 \\ \vdots & \ddots & 0 & *-\lambda & \theta & 0 & 0 & \vdots \\ \vdots & & \vdots & * & *-\lambda & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * & * & \ddots & \ddots & 0 \\ * & \cdots & * & * & * & \ddots & \ddots & \theta \\ * & \cdots & * & * & * & \cdots & * & *-\lambda \\ 0 & \cdots & 0 & \theta_{\underline{g}} & 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{82}$$

Now, we show that $rankD = n$ for two cases: (i) $\lambda = 0$ and (ii) $\lambda \neq 0$.

(i) Substituting $\lambda = 0$ into (82) yields (81). Removing the lowest two rows of D in (81) yields a square submatrix whose determinant is equal to the product of all the diagonal elements $\theta^n \neq 0$ because of condition (ii) in Definition 7. Therefore, it follows that $rankD = n$.

(ii) Consider the case of $\lambda \neq 0$. For D in (82), let all the rows between the second row and the $\underline{g}$th row be moved upward by one row, and at the same time let the first row be moved to the lowest row and also let the lowest row be moved to the $\underline{g}$th row. Then, D in (82) is written as follows:

$$\begin{array}{r} \\ \\ \\ \underline{g}\rightarrow \\ D= \end{array}\begin{bmatrix} -\lambda & \ddots & \ddots & \ddots & & & \vdots & \vdots \\ 0 & \ddots & \theta & * & \cdots & \cdots & * & \vdots \\ \vdots & \ddots & -\lambda & \theta & * & \cdots & * & 0 \\ 0 & \cdots & 0 & \theta_{\underline{g}} & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & 0 & *-\lambda & \theta & 0 & 0 & \vdots \\ \vdots & & \vdots & * & *-\lambda & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * & * & \ddots & \ddots & 0 \\ * & \cdots & * & * & * & \ddots & \ddots & \theta \\ * & \cdots & * & * & * & \cdots & * & *-\lambda \\ \theta & * & \cdots & \cdots & & \cdots & * & 0 \end{bmatrix}. \tag{83}$$

The determinant of the square submatrix obtained by eliminating the lowest two rows of D in (83) is equal to $(-\lambda)^{\underline{g}-1} \times \theta^{n-\underline{g}+1} \neq 0$. Therefore, it follows that $rankD = n$. From (i) and (ii), it follows that $rankD = n$ for any λ and $*$.

[II]: When $\underline{g} = n$, we obtain the following matrix $\tilde{D}$ by eliminating C_2 from D in (80):

$$\tilde{D} = \begin{bmatrix} \theta & * & * & \cdots & & * \\ -\lambda & \theta & * & & & \vdots \\ 0 & -\lambda & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \theta & * \\ \vdots & & & \ddots & -\lambda & \theta \\ 0 & \cdots & \cdots & \cdots & 0 & -\lambda \end{bmatrix}. \tag{84}$$

In the case of $\lambda = 0$, the determinant of the square submatrix obtained by eliminating the lowest row of $\tilde{D}$ in (84) is equal to $\theta^n \neq 0$. In the case of $\lambda \neq 0$, it is clear that the determinant of the square submatrix obtained by eliminating the first row of $\tilde{D}$ in (84) is equal to $(-\lambda)^n \neq 0$. Consequently, it follows that $rank\tilde{D} = n$ for every λ and $*$.

[III]: In the case of $\underline{g} = 0$, we obtain the following matrix $\tilde{D}$ by eliminating C_2 from D in (80):

$$\tilde{D} = \begin{bmatrix} \theta & 0 & 0 & \cdots & & 0 \\ -\lambda & \theta & 0 & & & \vdots \\ * & -\lambda & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \theta & 0 \\ \vdots & & & \ddots & -\lambda & \theta \\ * & \cdots & \cdots & \cdots & * & -\lambda \end{bmatrix}. \tag{85}$$

The determinant of the square submatrix obtained by eliminating the lowest row of $\tilde{D}$ in (85) is equal to $\theta^n \neq 0$. Therefore, it follows that $rank\tilde{D} = n$ for every λ and $*$.

From [I], [II], and [III], it has been proved that if all the conditions of Theorem 1 are satisfied, then Σ_P is observability-invariant.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for detailed comments on the paper which helped to improve the clarity of the manuscript.

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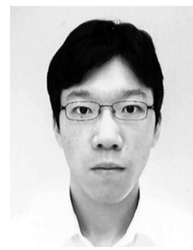

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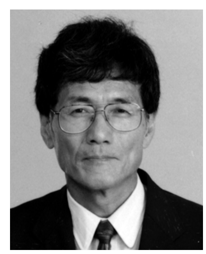

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