Probabilistic Constrained Model Predictive Control for Linear Discrete-time Systems with Additive Stochastic Disturbances

Tomoaki Hashimoto

***Abstract*— Model predictive control (MPC) is a kind of optimal feedback control in which the control performance over a finite future is optimized and its performance index has a moving initial time and a moving terminal time. The objective of this study is to propose a design method of MPC for linear discrete-time systems with stochastic disturbances under probabilistic constraints. For this purpose, the two-sided Chebyshev's inequality is applied to successfully handle probabilistic constraints with less computational load. A necessary and sufficient condition for the feasibility of the stochastic MPC is shown here. Moreover, a sufficient condition for the stability of the closed-loop system with stochastic MPC is derived by means of a linear matrix inequality.**

I. Introduction

The design methods of robust Model Predictive Control (MPC) against uncertain disturbances can be classified into deterministic approach and probabilistic approach. In the deterministic setting, most works are based on the min-max approach, where a performance index is minimized over the worst possible disturbance realization [2]-[4]. However, min-max approaches are often computationally demanding, and the control performance is often too conservative because no statistical properties of the disturbance are taken into account.

The other approach is addressed by stochastic MPC where expected values of performance indices, probabilistic constraints and convergence in probability are considered by exploiting the statistical information on the disturbance. In the deterministic MPC, the so-called hard constraints that must hold with probability 1 are taken into account for optimization problem. On the other hand, the stochastic MPC deals with the so-called soft constraints that cannot be fulfilled surely but with a given probability. It is known that a small relaxation of the probability requirement sometimes can lead to a significant improvement in the achievable control performance.

However, probabilistic constraints are generally intractable in an optimization problem. In recent decades, much attention has been paid to this difficulty of the stochastic MPC problem. Thus, several tractable methods haven been proposed to handle probabilistic constraints.

In [5]-[6], a second-order cone approximation method was proposed based on results from robust optimization to solve the stochastic linear-quadratic control problem. In [7], a stochastic MPC with feasibility and stability guarantees was proposed while considering the probabilistic polytopic sets instead of the deterministic bounds of uncertain disturbances. Also, the concept of probabilistic invariance was considered for the case of multiplicative uncertainty [8] and the case of both additive and multiplicative uncertainty [9]. On the one hand, a method for the convex approximation of probabilistic constraints with polytopic constraint functions was proposed in [10]. In [11], a decomposition method of soft constraints was proposed to obtain a lower bound to the convex optimization problem. Although the aforementioned papers [5]-[11] have achieved tremendous progress in dealing with probabilistic constraints of the stochastic MPC, there are several restrictions on the probability distributions of stochastic disturbances such as the normal (Gaussian) distribution, known distribution, finite-support and time-invariance.

On the other hand, the method proposed here enables us to address arbitrarily unknown probability distributions including non-Gaussian, infinitely-supported and time-variant distributions. The sampling methods using scenario approximation [12]-[13] and Bernstein approximation [14] are alternative methods for dealing with arbitrarily probability distributions. However, the sampling methods usually require heavy computational load. The objective in this study is to provide a stochastic MPC method for successfully dealing with probabilistic constraints with less computational load. For this purpose, we apply here the Chebyshev's inequality to transform soft constraints on the state variables into hard constraints on the control inputs. In [15], an ellipsoid approximation method based on the Chebyshev's inequality was proposed to handle soft constraints. However, the calculation of maximum volume inscribed ellipsoid is also computationally demanding and the stability of the closed-loop system was not discussed in [15]. In contrast, this paper provides a direct componentwise comparison method in the multi-dimensional Chebyshev's inequality to address soft constraints without using conservative approximation. Moreover, this study provides a necessary and sufficient condition for the feasibility of the stochastic MPC and a sufficient condition for the stability of the closed-loop system under several assumptions.

This paper is organized as follows. In Sec. II, we introduce some notations and define the system model. In Sec. III, we provide some preliminary results that are useful to construct the main results. The stochastic MPC problem is formulated in Sec. IV. The feasibility and stability of the proposed stochastic MPC method are discussed in Sec. V and Sec. VI, respectively. Finally, some concluding remarks are given in Sec. VII.

T. Hashimoto is with the Department of Systems Innovation, Graduate School of Engineering Science, Osaka University, 1-3 Machikaneyama, Toyonaka, Osaka 560-8531, Japan. Corresponding author's email: info@thashi.net or thashinet@gmail.com

II. NOTATION AND SYSTEM MODEL

Let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real numbers and natural numbers, respectively. Let $\mathbb{R}_+$ denote the set of nonnegative real numbers. For a matrix A, the transpose of A is denoted by A'. For matrices $A = \{a_{i,j}\}$ and $B = \{b_{i,j}\}$, let the inequalities between A and B such as $A > B$ and $A \geq B$ indicate that they are satisfied componentwisely, i.e., $a_{i,j} > b_{i,j}$ and $a_{i,j} \geq b_{i,j}$ hold for all i and j, respectively. Likewise, let each notation for the absolute value $|A|$, the square root $\sqrt{A}$ and the multiplication $A \otimes B$ indicate that it holds componentwisely, i.e., $|A| = \{|a_{i,j}|\}$, $\sqrt{A} = \{\sqrt{a_{i,j}}\}$ and $A \otimes B = \{a_{i,j} \times b_{i,j}\}$ for all i and j. Let $A \succ 0$ indicate that A is a positive definite matrix, i.e., $x'Ax > 0$ for any $x \neq 0$. For a vector x, let the norms $\|x\|$ and $\|x\|_A$ be defined by $\|x\| := x'x$ and $\|x\|_A := x'Ax$, respectively, where $A \succ 0$.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathbb{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathbb{K}_\infty$ if $\alpha \in \mathbb{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$.

Let the triple $(\Omega, \mathcal{F}, \mathcal{P})$ denote a probability space, where $\Omega \subseteq \mathbb{R}$ is the sampling space, $\mathcal{F}$ is the σ-algebra and $\mathcal{P}$ is the probability measure [16]. Ω is non-empty and is not necessarily finite. $\mathcal{P}(E)$ denotes the probability that the event E occurs. If $\mathcal{P}(E) = 1$ we say that E occurs almost surely. For a random variable $z : \Omega \to \mathbb{R}$ defined on $(\Omega, \mathcal{F}, \mathcal{P})$, let the expected value and the variance of z be denoted by $\mathcal{E}(z)$ and $\mathcal{V}(z)$, respectively. For a random vector $z = [z_1, \cdots, z_n]'$ whose each component is a random variable $z_i : \Omega \to \mathbb{R}$ $(i = 1, \cdots, n)$ defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$, we also adopt the same notation $\mathcal{E}(z)$ and $\mathcal{V}(z)$ to denote $\mathcal{E}(z) = [\mathcal{E}(z_1), \cdots, \mathcal{E}(z_n)]'$ and $\mathcal{V}(z) = [\mathcal{V}(z_1), \cdots, \mathcal{V}(z_n)]'$ for notational simplicity.

Throughout this paper, we consider the following linear discrete-time system with stochastic disturbances:

$$x(t+1) = Ax(t) + Bu(t) + Cw(t), \tag{1}$$

where $t \in \mathbb{N}$ is the time step, $x(t) : \mathbb{N} \to \mathbb{R}^n$ is the state, $u(t) : \mathbb{N} \to \mathbb{R}^m$ is the control input and $w(t) : \mathbb{N} \to \mathbb{R}^\ell$ is the unknown stochastic disturbance. More precisely, for each component $w_i : \mathbb{N} \times \Omega \to \mathbb{R}$ of w, the random sequence $\{w_i(t) : t \in \mathbb{N}\}$ is a collection of random variables on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with a filtration $\{\mathcal{F}_t : t \in \mathbb{N}\}$ [16]. The system coefficients $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{n\times \ell}$ are all known constant matrices. The pair (A, B) is assumed to be controllable. We also assume that the initial state $x(0)$ is given and all components of the state $x(t)$ are deterministically observable. Thus, we suppose that $\mathcal{E}(x(t)) = x(t)$ and $\mathcal{V}(x(t)) = 0$ at the present time t.

Next, we introduce some assumptions on the properties of the stochastic disturbances.

Assumption 1: $w_i(t)$ and $w_j(t)$ are independent each other for all $i \neq j$ and $t \in \mathbb{N}$. Also, $w_i(t)$ and $w_j(k)$ are independent each other for all $t \neq k$ and $j \in \{1, \cdots, \ell\}$.
Assumption 1 implies that all random variables $w_i(t)$ for $i \in \{1, \cdots, \ell\}$ and $t \in \mathbb{N}$ are independent one another.

Assumption 2: $\mathcal{E}(w(t))$ and $\mathcal{V}(w(t))$ are assumed to be known for every time t.

Note that the probability distributions of the random variable w_i are not necessarily assumed to be known. On the other hand, the probability distributions are assumed to be known in [11] to transform soft constraints into hard constraints, using cumulative distribution function. In the case where the probability distributions are unknown, we need to estimate $\mathcal{E}(w(t))$ and $\mathcal{V}(w(t))$ by means of some statistic method such as Gaussian process regression [17].

Assumption 3: There exists a positive real constant δ such that

$$\|C\mathcal{E}(w(t))\|_A \leq \delta \|\mathcal{E}(x(t))\|_A \tag{2}$$

is satisfied for all $A \succ 0$ and $t \in \mathbb{N}$.
Note that $\mathcal{E}(w(t))$ is assumed to be bounded, but $w(t)$ itself may be unbounded. Assumption 3 is introduced to discuss the stability at the origin of the averaged system for (1).

Definition 1: System (1) is said to be almost surely asymptotically stable in the mean if the following condition is satisfied:

$$\mathcal{P}\left(\lim_{t\to\infty} \mathcal{E}(x(t)) = 0\right) = 1. \tag{3}$$

III. PRELIMINARIES

In this section, we provide some preliminary results that are useful to derive the main results. The inequality shown below is known as the two-sided Chebyshev's inequality.

Lemma 1 ([18]): For a given random variable w with a mean μ and variance σ^2, the following inequality holds for every $\kappa \geq 1$.

$$\mathcal{P}(|w - \mu| \geq \kappa\sigma) \leq \frac{1}{\kappa^2}. \tag{4}$$

The following lemma is well known as Lyapunov stability theory.

Lemma 2 ([19]): Consider a system $x(t+1) = f(x(t))$, where $x(t) : \mathbb{N} \to \mathbb{R}^n$, $f(x(t)) : \mathbb{R}^n \to \mathbb{R}^n$ and $f(0) = 0$. Suppose that there exist a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}_+$, class $\mathbb{K}_\infty$ functions α_1, α_2 and a positive definite function α_3 satisfying all the following conditions:

$$\begin{aligned} &V(x) \geq \alpha_1(\|x\|) \\ &V(x) \leq \alpha_2(\|x\|) \\ &V(f(x)) - V(x) \leq -\alpha_3(\|x\|) \end{aligned}$$

Then, the origin $x = 0$ is asymptotically stable.
The equivalence shown below is known as Schur complement.

Lemma 3: For given block matrices A, B and C, the followings are equivalent.

$$\begin{bmatrix} A & B \\ B' & C \end{bmatrix} \succ 0$$
$$\Leftrightarrow A \succ 0, \;\; C - B'A^{-1}B \succ 0$$

The following lemmas are fundamental properties of the matrix theory.

Lemma 4: For any $A \succ 0 \in \mathbb{R}^{n\times n}$ and $b, c \in \mathbb{R}^n$,

$$\pm 2b'Ac \leq b'Ab + c'Ac.$$

Lemma 5: For any nonsingular matrix A, $(A')^{-1} = (A^{-1})'$ and $A'A \succ 0$ hold true. For any positive definite matrix A, it is true that $A^{-1} \succ 0$ and there exists B such that $A = B'B$.

IV. Problem Statement

In this section, we formulate the stochastic MPC problem of system (1). The control input at each time t is determined so as to minimize the performance index given by

$$J := \phi[x(t+N)] + \sum_{k=t}^{t+N-1} L[x(k), u(k)]. \tag{5a}$$

Therein, $N \in \mathbb{N}$ denotes the length of prediction horizon. ϕ and L are defined by

$$\phi := \mathcal{E}[x(t+N)'Px(t+N)], \tag{5b}$$
$$L := \mathcal{E}[x(k)'Qx(k)] + u(k)'Ru(k), \tag{5c}$$

where P, Q and R are positive definite constant matrices. $\phi \in \mathbb{R}_+$ is the terminal cost function and $L \in \mathbb{R}_+$ is the stage cost function over the prediction horizon.

Let $x_{min}(t)$ and $x_{max}(t) : \mathbb{N} \to \mathbb{R}^n$ denote the lower and upper bounds of $x(t)$, respectively. Let $p(t) = [p_1(t), \cdots, p_n(t)]'$: $\mathbb{N} \to [0\ 1]^n$ denote the probability in the vector form that means each component $p_i(t)$ belongs to $[0\ 1]$ for each time t.

Here, we impose the following probabilistic constraints on the optimization problem. For $k = t+1, \cdots, t+N$ and $i = 1, \cdots, n$,

$$\mathcal{P}\left(x_{i\,\min}(k) \le x_i(k) \le x_{i\,\max}(k)\right) \ge p_i(k), \tag{6}$$

where $x_{\min}(k)$, $x_{\max}(k)$ and $p(k)$ for $k = t+1, \cdots, t+N$ are given constant sequences. Condition (6) indicates that the state x_i over the prediction horizon must remain within the bound $[x_{i\,\min}\ \ x_{i\,\max}]$ at least with probability p_i.

For notational convenience, let $\mathbf{X} \in \mathbb{R}^{nN}$, $\mathbf{U} \in \mathbb{R}^{mN}$, $\mathbf{W} \in \mathbb{R}^{\ell N}$, $\mathbf{A} \in \mathbb{R}^{nN\times n}$, $\mathbf{B} \in \mathbb{R}^{nN\times mN}$, $\mathbf{C} \in \mathbb{R}^{nN\times \ell N}$, $\mathbf{Q} \in \mathbb{R}^{nN\times nN}$, $\mathbf{R} \in \mathbb{R}^{mN\times mN}$, $\mathbf{p} \in \mathbb{R}^{nN}$, $\mathbf{X}_{\min} \in \mathbb{R}^{nN}$ and $\mathbf{X}_{\max} \in \mathbb{R}^{nN}$ be defined by

$$\mathbf{X}(t) := \begin{bmatrix} x(t+1) \\ \vdots \\ x(t+N) \end{bmatrix}, \quad \mathbf{U}(t) := \begin{bmatrix} u(t) \\ \vdots \\ u(t+N-1) \end{bmatrix},$$

$$\mathbf{W}(t) := \begin{bmatrix} w(t) \\ \vdots \\ w(t+N-1) \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix},$$

$$\mathbf{B} := \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix},$$

$$\mathbf{C} := \begin{bmatrix} C & 0 & \cdots & 0 \\ AC & C & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}C & A^{N-2}C & \cdots & C \end{bmatrix},$$

$$\mathbf{Q} := \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & Q & 0 \\ 0 & \cdots & 0 & P \end{bmatrix}, \quad \mathbf{R} := \begin{bmatrix} R & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R \end{bmatrix}$$

$$\mathbf{p}(t) = \begin{bmatrix} p(t+1) \\ \vdots \\ p(t+N) \end{bmatrix}, \quad \mathbf{X}_{\min}(t) := \begin{bmatrix} x_{\min}(t+1) \\ \vdots \\ x_{\min}(t+N) \end{bmatrix},$$

$$\mathbf{X}_{\max}(t) := \begin{bmatrix} x_{\max}(t+1) \\ \vdots \\ x_{\max}(t+N) \end{bmatrix}.$$

Using the above notation, the performance index in (5) can be rewritten as follows:

$$J[x(t), \mathbf{X}(t), \mathbf{U}(t)] = \mathcal{E}[x(t)'Qx(t)] + \mathcal{E}[\mathbf{X}(t)'\mathbf{Q}\mathbf{X}(t)] + \mathbf{U}(t)'\mathbf{R}\mathbf{U}(t), \tag{7}$$

Note that the present state $x(t)$ is a deterministic vector. Hence, $\mathcal{E}(x(t)) = x(t)$. Then, probabilistic constraint (6) can be rewritten in the vector form as

$$\mathcal{P}\left(\mathbf{X}_{\min}(t) \le \mathbf{X}(t) \le \mathbf{X}_{\max}(t)\right) \ge \mathbf{p}(t). \tag{8}$$

More precisely, using the components $\mathbf{X}_{i\,\min}, \mathbf{X}_i, \mathbf{X}_{i\,\max} \in \mathbb{R}$ and $\mathbf{p}_i \in [0\ \ 1]$ of the vectors, condition (8) can be described as

$$\bigwedge_{i=1}^{nN} \left\{\mathcal{P}\left(\mathbf{X}_{i\,\min}(t) \le \mathbf{X}_i(t) \le \mathbf{X}_{i\,\max}(t)\right) \ge \mathbf{p}_i(t)\right\},$$

where the notation $\wedge$ denotes the logical conjunction.

Here, note that system (1) over the prediction horizon can be rewritten as

$$\mathbf{X}(t) = \mathbf{A}x(t) + \mathbf{B}\mathbf{U}(t) + \mathbf{C}\mathbf{W}(t). \tag{9}$$

Now, the optimal control problem can be formulated as the minimization problem of (7) subject to constraints (8) and (9) for given $x(t)$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{X}_{\min}(t)$, $\mathbf{X}_{\max}(t)$ and $\mathbf{p}(t)$.

V. Feasibility of Stochastic MPC

In this section, we provide a method for solving the stochastic MPC problem discussed in the previous section. It is shown here that the minimization of (7) subject to constraints (8) and (9) can be reduced to the quadratic programming subject to the deterministic constraints.

From (9), $\mathcal{E}(\mathbf{X}(t))$ and $\mathcal{V}(\mathbf{X}(t))$ are given by

$$\mathcal{E}(\mathbf{X}(t)) = \mathbf{A}x(t) + \mathbf{B}\mathbf{U}(t) + \mathbf{C}\mathcal{E}(\mathbf{W}(t)), \tag{10a}$$
$$\mathcal{V}(\mathbf{X}(t)) = \mathbf{C} \otimes \mathbf{C}\mathcal{V}(\mathbf{W}(t)), \tag{10b}$$

Substituting (10a) into (7) and neglecting the terms that don't contain $\mathbf{U}(t)$, we have

$$\min_{\mathbf{U}(t)} J[x(t), \mathbf{X}(t), \mathbf{U}(t)] = \tag{11}$$
$$\min_{\mathbf{U}(t)} \left\{ \begin{array}{l} \mathbf{U}'(t)\left(\mathbf{B}'\mathbf{Q}\mathbf{B}+\mathbf{R}\right)\mathbf{U}(t) \\ \qquad +2\left(\mathbf{A}x(t)+\mathbf{C}\mathcal{E}(\mathbf{W}(t))\right)'\mathbf{Q}\mathbf{B}\mathbf{U}(t) \end{array} \right\}.$$

Note that the minimization problem of J has been reduced to the quadratic programming with respect to $\mathbf{U}$. In general, however, it is not straightforward to solve the quadratic programming with probabilistic constraint (8). In the following, we convert the probabilistic constraint into the deterministic one, using the Chebyshev's inequality [18]. For this purpose, we now state the theorem that plays an important role in this study.

Theorem 1: Suppose that the following condition holds:

$$\mathbf{U}_{\min}(t) \le \mathbf{B}\mathbf{U}(t) \le \mathbf{U}_{\max}(t), \tag{12}$$

where $\mathbf{U}_{\min}$ and $\mathbf{U}_{\max}$ are defined by

$$\mathbf{U}_{\min}(t) := \mathbf{X}_{\min}(t) + \boldsymbol{\kappa}(t) \otimes \sqrt{\mathbf{C}\otimes\mathbf{C}\mathcal{V}(\mathbf{W}(t))} - \mathbf{A}x(t) - \mathbf{C}\mathcal{E}(\mathbf{W}(t)), \tag{13a}$$

$$\mathbf{U}_{\max}(t) := \mathbf{X}_{\max}(t) - \boldsymbol{\kappa}(t) \otimes \sqrt{\mathbf{C}\otimes\mathbf{C}\mathcal{V}(\mathbf{W}(t))} - \mathbf{A}x(t) - \mathbf{C}\mathcal{E}(\mathbf{W}(t)). \tag{13b}$$

$$\boldsymbol{\kappa}(t) := \left[\frac{1}{\sqrt{1-\mathbf{p}_1(t)}}, \cdots, \frac{1}{\sqrt{1-\mathbf{p}_{nN}(t)}}\right]'. \tag{13c}$$

Then, probabilistic condition (8) is fulfilled.

Proof: Using Lemma 1, we have the following inequality in the componentwise form:

$$\mathcal{P}\left(|\mathbf{X}_i(t) - \mathcal{E}(\mathbf{X}_i(t))| < \boldsymbol{\kappa}_i(t)\sqrt{\mathcal{V}(\mathbf{X}_i(t))}\right) \ge 1 - \frac{1}{\boldsymbol{\kappa}_i^2(t)}.$$

Using (13c), we can rewrite the above inequality as the one in the vector form.

$$\mathcal{P}\left(|\mathbf{X}(t) - \mathcal{E}(\mathbf{X}(t))| < \boldsymbol{\kappa}(t) \otimes \sqrt{\mathcal{V}(\mathbf{X}(t))}\right) \ge \mathbf{p}(t).$$

Here, it is important to note that if both the following conditions

$$\mathbf{X}_{\min}(t) \le \mathcal{E}(\mathbf{X}(t)) - \boldsymbol{\kappa} \otimes \sqrt{\mathcal{V}(\mathbf{X}(t))} \tag{14a}$$

$$\mathcal{E}(\mathbf{X}(t)) + \boldsymbol{\kappa} \otimes \sqrt{\mathcal{V}(\mathbf{X}(t))} \le \mathbf{X}_{\max}(t) \tag{14b}$$

are satisfied, then probabilistic condition (8) is fulfilled. Substituting (10) into (14), we can see that condition (14) is equivalent to condition (12). Therefore, we can conclude that if deterministic constraint (12) on $\mathbf{U}(t)$ is satisfied, then probabilistic constraint (8) on $\mathbf{X}(t)$ is also satisfied. This completes the proof. ■

Remark 1: By Theorem 1, the minimization problem of (11) with probabilistic constraint (8) can be reduced to the quadratic programming with deterministic constraint (12) that can be solved using a conventional algorithm [20].

Next, we apply the so-called Gale's theorem to derive the feasibility condition for inequality constraint (12).

Corollary 1: For given $\mathbf{B}$, $\mathbf{U}_{\min}$ and $\mathbf{U}_{\max}$, exactly one of the following statements holds:

I) there exists $\mathbf{U}$ such that constraint (12) is satisfied;

II) there exsit $\boldsymbol{\lambda} \in \mathbb{R}^{nN}$ and $\boldsymbol{\rho} \in \mathbb{R}^{nN}$ such that

$$\boldsymbol{\lambda} \ge 0, \quad \boldsymbol{\lambda}'\mathbf{B} = 0, \quad \boldsymbol{\lambda}'\mathbf{U}_{\min} > 0, \tag{15a}$$

$$\boldsymbol{\rho} \ge 0, \quad \boldsymbol{\rho}'\mathbf{B} = 0, \quad \boldsymbol{\rho}'\mathbf{U}_{\max} < 0 \tag{15b}$$

are satisfied.

Proof: Let $\mathbf{F}$ and $\mathbf{G}$ be defined by

$$\mathbf{F} = \begin{bmatrix} -\mathbf{B} \\ \mathbf{B} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} -\mathbf{U}_{\min} \\ \mathbf{U}_{\max} \end{bmatrix}.$$

Then, constraint (12) can be reduced to the following:

$$\mathbf{F}\begin{bmatrix} \mathbf{U} \\ \mathbf{U} \end{bmatrix} \le \mathbf{G}. \tag{16}$$

Applying Theorem 22.1 in [21] to (16), we can see that exactly one of the following two statements (i) or (ii) holds:

i) there exists $\mathbf{U}$ such that (16) is satisfied;

ii) there exsit $\boldsymbol{\lambda} \in \mathbb{R}^{nN}$ and $\boldsymbol{\rho} \in \mathbb{R}^{nN}$ such that

$$\begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\rho} \end{bmatrix} \ge 0, \quad \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\rho} \end{bmatrix}'\mathbf{F} = 0, \quad \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\rho} \end{bmatrix}'\mathbf{G} < 0 \tag{17}$$

are satisfied.

It is obvious that (i) and (ii) are equivalent to (I) and (II), respectively. Therefore, the proof is completed. ■

Corollary 1 implies that both statements (I) and (II) cannot hold simultaneously, that is, one of (I) and (II) is always valid. Hence, we can see that there exists $\mathbf{U}$ such that (12) is satisfied if and only if either $\boldsymbol{\lambda}$ satisfying (15a) or $\boldsymbol{\rho}$ satisfying (15b) doesn't exist. Therefore, we have the following statement.

Corollary 2: There exists $\mathbf{U}$ such that (12) is satisfied if and only if $[\boldsymbol{\lambda}'\ \boldsymbol{\rho}'] = 0$ is the only solution of $[\boldsymbol{\lambda}'\ \boldsymbol{\rho}'] \ge 0$, $[\boldsymbol{\lambda}'\ \boldsymbol{\rho}']\mathbf{F} = 0$, $[\boldsymbol{\lambda}'\ \boldsymbol{\rho}']\mathbf{G} \le 0$.

VI. Stability of Stochastic MPC

In this section, we study the stability of the closed-loop system with the stochastic MPC. We employ here the Lyapunov stability theory to derive a sufficient condition for the asymptotic stability in the mean of the stochastic MPC system. Thus, the performance index should be chosen as the Lyapunov function to guarantee the asymptotic stability in the mean. Hence, in the subsequent discussion, we consider the cost functions ϕ, L as follows:

$$\phi = \mathcal{E}(x(t+N))'P\mathcal{E}(x(t+N)),$$
$$L = \mathcal{E}(x(k))'Q\mathcal{E}(x(k)) + u(k)'Ru(k)$$

Note that the minimization problem of the above cost functions can be reduced to the same minimization problem in (11). Therefore, the stability of MPC in performance index (11) is equivalent to the stability of MPC in the above performance index.

First, we consider the existence of the control input $u(t) = K\mathcal{E}(x(t))$ such that the following inequality holds, where $K \in \mathbb{R}^{m\times n}$ is a constant matrix.

$$\phi[\mathcal{E}(x(t+1))] - \phi[\mathcal{E}(x(t))] \le -L[\mathcal{E}(x(t)), u(t)] \tag{18}$$

Recall that P, Q and R are weighting matrices introduced in (5). Let Z, G and M be matrices such that $Z = P^{-1}$, $G = KZ$ and $M'M = 2\delta P + Q$, where δ is a positive constant satisfying (2). Since $2\delta P + Q$ is a positive definite, we see from Lemma 5 that there always exists M defined above.

The following lemma plays an important role to establish the stability criteria for the closed-loop system using the stochastic MPC.

Lemma 6: Inequality (18) is satisfied if there exist Z, G and M such that the following linear matrix inequality holds:

$$\begin{bmatrix} Z & ZA' + G'B' & ZM' & G'R' \\ AZ + BG & \frac{Z}{2} & 0 & 0 \\ MZ & 0 & I & 0 \\ RG & 0 & 0 & R \end{bmatrix} \succ 0. \quad (19)$$

Proof: It is straightforward that

$$\begin{aligned} &\phi[\mathcal{E}(x(t+1))] - \phi[\mathcal{E}(x(t))] = \mathcal{E}(w(t))' \{C'PC\} \mathcal{E}(w(t)) \\ &+ \mathcal{E}(x(t))' \{(A+BK)'P(A+BK) - P\} \mathcal{E}(x(t)) \\ &+ 2\mathcal{E}(x(t))'(A+BK)'PC\mathcal{E}(w(t)). \end{aligned} \quad (20)$$

Applying Lemma 4 to the last term in the right-hand side of (20) yields

$$\begin{aligned} &\phi[\mathcal{E}(x(t+1))] - \phi[\mathcal{E}(x(t))] \leq 2\mathcal{E}(w(t))' \{C'PC\} \mathcal{E}(w(t)) \\ &+ \mathcal{E}(x(t))' \{2(A+BK)'P(A+BK) - P\} \mathcal{E}(x(t)). \end{aligned} \quad (21)$$

Furthermore, applying Assumption 3 to the first term in the right-hand side of (21) yields

$$\begin{aligned} &\phi[\mathcal{E}(x(t+1))] - \phi[\mathcal{E}(x(t))] \leq \\ &\mathcal{E}(x(t))' \{2\delta P + 2(A+BK)'P(A+BK) - P\} \mathcal{E}(x(t)). \end{aligned} \quad (22)$$

Noting that

$$L = \mathcal{E}(x(t))'(Q + K'RK)\mathcal{E}(x(t)), \quad (23)$$

we can see that if

$$P - 2(A+BK)'P(A+BK) - 2\delta P - Q - K'RK \succ 0 \quad (24)$$

is satisfied, then inequality (18) holds true.

In the following, it is shown that above inequality (24) is equivalent to inequality (19).

Pre- and post-multiplying (24) by Z yields

$$Z - 2(ZA' + G'B')Z^{-1}(AZ + BG) - ZM'MZ - G'RG \succ 0 \quad (25)$$

Using the following relation

$$ZM'MZ - G'RG = \begin{bmatrix} MZ \\ RG \end{bmatrix}' \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix}^{-1} \begin{bmatrix} MZ \\ RG \end{bmatrix},$$

we can see that (25) is equivalent to the following:

$$Z - \begin{bmatrix} AZ + BG \\ MZ \\ RG \end{bmatrix}' \begin{bmatrix} \frac{Z}{2} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & R \end{bmatrix}^{-1} \begin{bmatrix} AZ + BG \\ MZ \\ RG \end{bmatrix} \succ 0 \quad (26)$$

Using Lemma 3, we can see that the Schur compliment of the (1, 1) block of (19) is equivalent to (26). Consequently, the proof has been completed. ∎

Let a function $V[\mathcal{E}(x(t))] : \mathbb{R}^n \to \mathbb{R}_+$ be defined by

$$V[\mathcal{E}(x(t))] := \min_{\mathbf{U}(t)} J[\mathcal{E}(x(t)), \mathcal{E}(\mathbf{X}(t)), \mathbf{U}(t)]. \quad (27)$$

Let $\mathbf{U}^*(t)$ denote the sequence of the optimal control input over the prediction horizon defined by

$$\begin{aligned} \mathbf{U}^*(t) &:= \begin{bmatrix} u^*(t) \\ \vdots \\ u^*(t+N-1) \end{bmatrix} \\ &:= \arg\min_{\mathbf{U}(t)} J[\mathcal{E}(x(t)), \mathcal{E}(\mathbf{X}(t)), \mathbf{U}(t)]. \end{aligned} \quad (28)$$

Let $\mathbf{X}^*(t) = [x^*(t+1), \cdots, x^*(t+N)]'$ denote the optimal state sequence of the closed-loop system over the prediction horizon using $\mathbf{U}^*(t)$. Let $\hat{\mathbf{U}}^*(t+1)$ be defined by

$$\hat{\mathbf{U}}^*(t+1) := \begin{bmatrix} u^*(t+1) \\ \vdots \\ u^*(t+N-1) \\ u(t+N) \end{bmatrix}. \quad (29)$$

Therein, the final optimal control input $u^*(t+N)$ is replaced with any feasible control input $u(t+N)$. Accordingly, let $\hat{\mathbf{X}}^*(t+1)$ be the state sequence of the closed-loop system using $\hat{\mathbf{U}}^*(t+1)$.

Here, we introduce the well-known standard assumption for the stability analysis of the MPC system [19].

Assumption 4: There exists a function $\alpha \in \mathbb{K}_\infty$ such that

$$V[\mathcal{E}(x(t))] \leq \alpha\left(\|\mathcal{E}(x(t))\|\right) \quad (30)$$

is satisfied for all $t \in \mathbb{N}$.

Note that if there exists a positive constant ρ such that

$$\|u^*(t)\| \leq \rho \|\mathcal{E}(x(t))\|$$

is satisfied for all $t \in \mathbb{N}$, then Assumption 4 is satisfied. Thereby, Assumption 4 is called the weak controllability assumption [19].

Assumption 5: There exist $\mathbf{U}^*(t)$ and $\hat{\mathbf{U}}^*(t+1)$ that satisfy constraint (12) for all $t \in \mathbb{N}$.

Here, we provide the stability criteria for the closed-loop system using the stochastic MPC.

Theorem 2: Under Assumptions 1–5, the closed-loop system using stochastic MPC input $\mathbf{U}^*(t)$ is almost surely asymptotically stable in the mean if there exist Z, G and M such that linear matrix inequality (19) is satisfied.

Proof: It follows from (27) that

$$\begin{aligned} &V[\mathcal{E}(x(t))] = L\left[\mathcal{E}\left(x(t)\right), u^*(t)\right] \\ &+ \sum_{k=t+1}^{t+N-1} L\left[\mathcal{E}\left(x^*(k)\right), u^*(k)\right] + \phi\left[\mathcal{E}\left(x^*(t+N)\right)\right] \end{aligned} \quad (31)$$

Using the relation

$$\begin{aligned} &J\left[\mathcal{E}(x(t+1)), \mathcal{E}(\mathbf{X}^*(t+1)), \mathbf{U}^*(t+1)\right] \\ &\quad \leq J\left[\mathcal{E}(x(t+1)), \mathcal{E}(\hat{\mathbf{X}}^*(t+1)), \hat{\mathbf{U}}^*(t+1)\right], \end{aligned} \quad (32)$$

we have the following:

$$
\begin{aligned}
V[\mathcal{E}(x(t+1))] = &\sum_{k=t+1}^{t+N} L\left[\mathcal{E}\left(x^*(k)\right), u^*(k)\right] \\
&\qquad + \phi\left[\mathcal{E}\left(x^*(t+N+1)\right)\right] \\
\le &\sum_{k=t+1}^{t+N-1} L\left[\mathcal{E}\left(x^*(k)\right), u^*(k)\right] \\
&+ L\left[\mathcal{E}\left(x^*(t+N)\right), u(t+N)\right] + \phi\left[\mathcal{E}\left(x(t+N+1)\right)\right] \\
=: &\hat{V}[\mathcal{E}(x(t+1))] \qquad (33)
\end{aligned}
$$

Let $\hat{V}[\mathcal{E}(x(t+1))]$ be defined as above. Using the above inequality, we have the following:

$$
\begin{aligned}
&V[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \\
&\le \hat{V}[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \\
&= -L\left[\mathcal{E}\left(x(t)\right), u^*(t)\right] + L\left[\mathcal{E}\left(x^*(t+N)\right), u(t+N)\right] \\
&+ \phi\left[\mathcal{E}\left(x(t+N+1)\right)\right] - \phi\left[\mathcal{E}\left(x^*(t+N)\right)\right] \qquad (34)
\end{aligned}
$$

We can see from Lemma 6 that there exists $u(t+N)$ such that the following inequality holds.

$$
\begin{aligned}
&\phi\left[\mathcal{E}\left(x(t+N+1)\right)\right] - \phi\left[\mathcal{E}\left(x^*(t+N)\right)\right] \\
&\le -L\left[\mathcal{E}\left(x^*(t+N)\right), u(t+N)\right] \qquad (35)
\end{aligned}
$$

Applying (35) to (34) yields

$$
V[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \le -L\left[\mathcal{E}\left(x(t)\right), u^*(t)\right]. \qquad (36)
$$

Here, note that there exists a positive constant ν such that the following inequalities hold.

$$
\begin{aligned}
V[\mathcal{E}(x(t))] &\ge L\left[\mathcal{E}\left(x(t)\right), u^*(t)\right] \\
&\ge \mathcal{E}(x(t))' Q \mathcal{E}(x(t)) \\
&\ge \nu \left\|\mathcal{E}(x(t))\right\| \qquad (37)
\end{aligned}
$$

Therefore, it follows that

$$
V[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \le -\nu \left\|\mathcal{E}(x(t))\right\| \qquad (38)
$$

Consequently, under Assumption 4, we can see that there exist $\mathbb{K}_\infty$ functions α_1 and α_2 such that the following inequalities are satisfied.

$$
\begin{aligned}
\alpha_1\left(\left\|\mathcal{E}\left(x(t)\right)\right\|\right) \le V[\mathcal{E}(x(t))] \le \alpha_2\left(\left\|\mathcal{E}\left(x(t)\right)\right\|\right) \\
V[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \le -\alpha_1\left(\left\|\mathcal{E}\left(x(t)\right)\right\|\right)
\end{aligned}
$$

Hence, using Lemma 2, we can conclude that $\mathcal{E}(x(t)) = 0$ is asymptotically stable. This completes the proof. ∎

Remark 2: From Theorem 2, we can verify the stability of the closed loop system with the stochastic MPC by checking linear matrix inequality (19) that is solvable using a conventional algorithm [22].

VII. CONCLUSION

In this study, we proposed a design method of model predictive control (MPC) for linear discrete-time systems with stochastic disturbances under probabilistic constraints. The Chebyshev's inequality was applied to successfully handle probabilistic constraints with less computational load. Thus, the stochastic MPC problem with probabilistic constraints was reduced to the quadratic programming with deterministic constraints that can be solved using a conventional algorithm. Furthermore, we provide here a sufficient condition for the stability of the closed-loop system by means of a linear matrix inequality that can be easily verified using a conventional algorithm. To develop from the proposed method into the output-feedback based method and to consider general polytopic state constraints or mixed state-input constraints are possible future works.

REFERENCES

[1] D. Q. Mayne, J. B. Rawlings, C. V. Rao and P. O. M. Scokaert, Constrained model predictive control: Stability and optimality, *Automatica*, Vol. 36, 2000, pp.789-814.

[2] P. Scokaert and D. Mayne, Min-max Feedback Model Predictive Control for Constrained Linear Systems, *IEEE Trans. Automat. Contr.*, Vol. 43, 1998, pp.1136-1142.

[3] T. Alamo, D. Peña, D. Limon and E. Camacho, Constrained Minmax Predictive Control: Modifications of the Objective Function Leading to Polynomial Complexity, *IEEE Trans. Automat. Contr.*, Vol. 50, 2005, pp.710-714.

[4] D. Peña, T. Alamo, A. Bemporad and E. Camacho, A Decomposition Algorithm for Feedback Min-max Model Predictive Control, *IEEE Trans. Automat. Contr.*, Vol. 51, 2006, pp.1688-1692.

[5] D. Bertsimas and D. B. Brown, Constrained Stochastic LQC: A Tractable Approach, *IEEE Trans. Automat. Contr.*, Vol. 52, 2007, pp.1826-1841.

[6] P. Hokayema, E. Cinquemani, D. Chatterjee, F Ramponid and J. Lygeros, Stochastic receding horizon control with output feedback and bounded controls, *Automatica*, Vol. 48, 2012, pp.77-88.

[7] M. Cannon, B. Kouvaritakis and D. Ng, Probabilistic tubes in linear stochastic model predictive control, *Systems & Control Letters*, Vol. 58, 2009, pp.747-753.

[8] M. Cannon, B. Kouvaritakis and X. Wu, Model predictive control for systems with stochastic multiplicative uncertainty and probabilistic constraints, *Automatica*, Vol. 45, 2009, pp.167-172.

[9] M. Cannon, B. Kouvaritakis and X. Wu, Probabilistic Constrained MPC for Multiplicative and Additive Stochastic Uncertainty, *IEEE Trans. Automat. Contr.*, Vol. 54, 2009, pp.1626-1632.

[10] E. Cinquemani, M. Agarwal, D. Chatterjee and J. Lygeros, Convexity and convex approximations of discrete-time stochastic control problems with constraints, *Automatica*, Vol. 47, 2011, pp.2082-2087.

[11] M. Ono, L. Blackmore and B. C. Williams, Chance Constrained Finite Horizon Optimal Control with Nonconvex Constraints, *Proceedings of American Control Conference*, 2010, pp.1145-1152.

[12] G. C. Calafiore and M. C. Campi, The Scenario Approach to Robust Control Design, *IEEE Trans. Automat. Contr.*, Vol. 51, 2006, pp.742-753.

[13] J. Matuško and F. Borrelli, Scenario-Based Approach to Stochastic Linear Predictive Control, *Proceedings of the 51st IEEE Conference on Decision and Control*, 2012, pp.5194-5199.

[14] A. Nemirovski and A. Shapiro, Convex Approximations of Chance Constrained Programs, *SIAM J. Control Optim.*, Vol. 17, 2006, pp.969-996.

[15] Z. Zhou and R. Cogill, An Algorithm for State Constrained Stochastic Linear-Quadratic Control, *Proceedings of American Control Conference*, 2011, pp.1476-1481.

[16] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, Springer, 6th edition, 2010.

[17] C. E. Rasmussen and C. K. I. Williams, *Gaussian Processes for Machine Learning*, MIT Press, 2006. *IEEE Trans. Automat. Contr.*, Vol. 51, 2006, pp.742-753.

[18] W. Feller, *An Introduction to Probability Theory and Its Applications: Vol. 2*, Wiley, 2nd Edition, 1971.

[19] J. B. Rawlings and D. Q. Mayne, *Model Predictive Control: Theory and Design*, Nob Hill Publishing, 2009.

[20] J. Nocedal and S. J. Wright, Numerical optimization, *Springer Series in Operation Research and Financial Engineering*, Springer, 2006.

[21] R. T. Rockafellar, *Convex Analysis*, Princeton Press, 1997.

[22] L. E. Ghaoui and S.-I. Niculescu, *Advances in Linear Matrix Inequality Methods in Control*, Society for Industrial and Applied Mathematics, 1987.