Receding Horizon Control for a Class of Discrete-time Nonlinear Implicit Systems

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***Abstract*— Receding horizon control is a type of optimal feedback control in which control performance over a finite future is optimized with a performance index that has a moving initial time and terminal time. Implicit systems belong to a more generalized class of systems than a class of explicit systems, because they can additionally contain algebraic constraints. The objective of this study is to develop a novel design method of receding horizon control for a generalized class of discrete-time nonlinear implicit systems. Using the variational principle, we derive the stationary conditions that must be satisfied for a performance index to be optimized. Moreover, we provide numerical algorithms for solving the obtained stationary conditions. Next, we establish the stability criterion for the closed-loop system with the proposed method. Finally, the effectiveness of the proposed method is verified by numerical simulations.**

I. Introduction

Receding horizon control (RHC), also known as model predictive control, is a well-established control method in which the current control input is obtained by solving a finite-horizon open-loop optimal control problem using the current state of the system as the initial state, and this procedure is repeated at each sampling instant [1]. Thus, RHC is a type of optimal feedback control in which the control performance over a finite future is optimized with a performance index that has a moving initial time and terminal time. In general, a feedback controller is more robust against disturbance and model error than a feedforward controller. Moreover, an important advantage of RHC is its ability to deal with constraints on control inputs and states.

For the RHC problem of nonlinear explicit systems, the stationary conditions that must be satisfied for a performance index to be optimized are well-known as the Euler–Lagrange equations. Several numerical algorithms for solving the Euler–Lagrange equations have been proposed in [2]-[3]. For nonlinear implicit systems, H^∞ optimal control problems have been investigated in [4]-[5]. Thus, the local stability criterion around a neighborhood of the origin has been established in [4]-[5].

To the best of my knowledge, the RHC problem of nonlinear implicit systems is still an open problem as far as general classes of systems are concerned. The objective of this study is to solve the RHC problem of nonlinear implicit systems and to establish the global stability criterion for the closed-loop system with the proposed RHC.

In Section II, we define the system model and introduce some preliminary results. In Section III, we first derive the generalized Euler–Lagrange equations for the optimal control problem of nonlinear implicit systems. Next, we provide a brief description of the algorithm for numerically solving the obtained Euler–Lagrange equations. In Section IV, we first study the existence and uniqueness of the solution of the system equation. Next, we show the stability criterion for the closed-loop system with the proposed RHC. In Section V, we provide an illustrative example to verify the effectiveness of the proposed method. Finally, concluding remarks are stated in Section VI.

Notation: Let $\mathbb{R}$ denote a set of real numbers. Let $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the sets of nonnegative real numbers and integers, respectively. Let $\mathbb{N}$ denote the sets of natural numbers (positive integers). For a matrix $A \in \mathbb{R}^{n\times n}$, the transpose and the inverse of A are denoted by A^{T} and A^{-1}, respectively. The determinant and rank of a matrix A are denoted by $\det(A)$ and $\mathrm{rank}(A)$, respectively. Let I denote the identity matrix.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathbb{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathbb{K}_\infty$ if $\alpha \in \mathbb{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$.

For a scalar function $\phi(x) : \mathbb{R}^n \to \mathbb{R}$, the differentiation of $\phi(x)$ with respect to $x \in \mathbb{R}^n$ is defined by

$$\frac{\partial \varphi(x)}{\partial x} := \begin{bmatrix} \frac{\partial \varphi(x)}{\partial x_1} & \frac{\partial \varphi(x)}{\partial x_2} & \cdots & \frac{\partial \varphi(x)}{\partial x_n} \end{bmatrix}$$

The Jacobian matrix of a vector-valued function $F(x)$: $\mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\frac{\partial F(x)}{\partial x} := \begin{bmatrix} \frac{\partial F_1(x)}{\partial x_1} & \cdots & \frac{\partial F_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n(x)}{\partial x_1} & \cdots & \frac{\partial F_n(x)}{\partial x_n} \end{bmatrix}$$

II. System Model and Preliminary Results

In this study, we consider the following discrete-time nonlinear implicit systems:

$$E\left(x(t)\right) x(t+1) = F\left(x(t), u(t)\right), \tag{1}$$

where $t \in \mathbb{Z}_+$, $x(t) : \mathbb{Z}_+ \to \mathbb{R}^n$, and $u(t) : \mathbb{Z}_+ \to \mathbb{R}^m$ denote a temporal variable, the state, and the control input, respectively. $E\left(x(t)\right) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $F\left(x(t), u(t)\right)$: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuously differentiable function. For examples, the discretized equations for nonlinear diffusion process [6] and RLC network circuits [7] belong to a class of systems described by (1).

Here, we assume that the state $x(t)$ satisfying (1) is known for all $t \in \mathbb{Z}_+$. Moreover, we assume that $E\left(x(t)\right)$

This work was supported in part by Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientists (24760334).

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is not necessarily of full-rank, i.e., $\operatorname{rank} E\left(x(t)\right) \leq n$. In particular, we call system (1) the descriptor system when $\det E\left(x(t)\right) = 0$. Without loss of generality, we assume $F(0,0) = 0$, that is, the origin $x = 0$ is the equilibrium point.

Next, we introduce some preliminary results. The following lemma is well known as Lyapunov stability theory.

Lemma 1 ([1]): Consider a system $x(t+1) = F(x(t))$, where $x(t) : \mathbb{Z}_+ \to \mathbb{R}^n$, $F(x(t)) : \mathbb{R}^n \to \mathbb{R}^n$ and $F(0) = 0$. Suppose that there exist a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}_+$, class $\mathbb{K}_\infty$ functions α_1, α_2 and a positive definite function α_3 satisfying all the following conditions:

$$\begin{aligned} &V(x) \geq \alpha_1\left(\|x\|\right) \\ &V(x) \leq \alpha_2\left(\|x\|\right) \\ &V(F(x)) - V(x) \leq -\alpha_3\left(\|x\|\right) \end{aligned}$$

Then, the origin $x = 0$ is asymptotically stable.
The following lemma is well known as implicit function theorem.

Lemma 2 ([8]): Let $f(x,y) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be continuously differentiable function. For each point (x,y) of an open set $S \subset \mathbb{R}^n \times \mathbb{R}^m$, suppose that $f(x,y) = 0$ and the Jacobian matrix $\partial f(x,y)/\partial y$ is nonsingular. Then, there exist neighborhoods $W \subset \mathbb{R}^n$ of x and $U \subset \mathbb{R}^m$ of y such that for each $x \in W$ the equation $f(x,y) = 0$ has a unique solution $y \in U$. Moreover, this solution can be given as a continuously differentiable function $y = g(x)$.

III. Receding Horizon Control

In this section, we consider the receding horizon control problem of system (1). Using the variational principle, we analytically derive the stationary conditions that must be satisfied for a performance index to be optimized. The control input at each time t is determined so as to minimize the performance index given by

$$J = \phi\left(x(t+N)\right) + \sum_{k=t}^{t+N-1} L\left(x(k), u(k)\right). \tag{2}$$

Therein, $N \in \mathbb{N}$ denotes the length of prediction horizon. $\phi : \mathbb{R}^n \to \mathbb{R}_+$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ are so-called terminal cost function and stage cost function, respectively, and assumed to be continuously differentiable functions with $\phi(0) = 0$ and $L(0,0) = 0$.

The minimization problem of (2) subject to (1) can be reduced to the minimization of the following performance index introduced using the costate $\lambda \in \mathbb{R}^n$ associated with system equation (1):

$$\begin{aligned} \bar{J} = \phi\left(x(t+N)\right) + \sum_{k=t}^{t+N-1} \Big[& L\left(x(k), u(k)\right) \\ & + \lambda^{\mathrm{T}}(k+1)\left\{F\left(x(k), u(k)\right) - E\left(x(k)\right)x(k+1)\right\}\Big] \end{aligned} \tag{3}$$

Let $H \in \mathbb{R}$ denote the Hamiltonian defined by

$$H := L\left(x(k), u(k)\right) + \lambda^{\mathrm{T}}(k+1) F\left(x(k), u(k)\right). \tag{4}$$

Let $\delta\bar{J}$, δx, $\delta\lambda$, and δu denote the variations (infinitesimal changes) in $\bar{J}$, x, λ, and u, respectively. Since the optimal solution must satisfy the stationary condition $\delta\bar{J} = 0$, we need to consider the variation $\delta\bar{J}$ due to the variations δx, $\delta\lambda$, and δu. Then, we need to calculate $\delta\bar{J} = J\left(x+\delta x, \lambda+\delta\lambda, u+\delta u\right) - J\left(x, \lambda, u\right)$. Applying the Taylor expansion into $J\left(x+\delta x, \lambda+\delta\lambda, u+\delta u\right)$ around (x, λ, u) and neglecting the high order terms of each variation, we can compute the variation in $\bar{J}$. Thus, $\delta\bar{J}$ can be described by

$$\begin{aligned} \delta\bar{J} = & \sum_{k=t}^{t+N} \frac{\partial \bar{J}}{\partial x(k)} \delta x(k) + \sum_{k=t}^{t+N-1} \frac{\partial \bar{J}}{\partial u(k)} \delta u(k) \\ & + \sum_{k=t+1}^{t+N} \frac{\partial \bar{J}}{\partial \lambda(k)} \delta\lambda(k). \end{aligned} \tag{5}$$

In fact, the first term in the right-hand side of (5) consists of the following terms:

$$\begin{aligned} & \frac{\partial \varphi}{\partial x(t+N)} \delta x(t+N) + \sum_{k=t}^{t+N-1} \left\{ \frac{\partial H}{\partial x(k)} \delta x(k) \right. \\ & + \frac{\partial\left(\lambda^{\mathrm{T}}(k+1) E(x(k)) x(k+1)\right)}{\partial x(k)} \delta x(k) \\ & \left. + \frac{\partial\left(\lambda^{\mathrm{T}}(k+1) E(x(k)) x(k+1)\right)}{\partial x(k+1)} \delta x(k+1) \right\}. \end{aligned} \tag{6}$$

Here, note that the following relation is useful to rewrite (6) as a linear combination with respect to $\delta x(k)$.

$$\begin{aligned} & \sum_{k=t}^{t+N-1} -\lambda^{\mathrm{T}}(k+1) E\left(x(k)\right) \delta x(k+1) \\ & = -\lambda^{\mathrm{T}}(t+N) E\left(x(t+N-1)\right) \delta x(t+N) \\ & - \sum_{k=t}^{t+N-2} \lambda^{\mathrm{T}}(k+1) E\left(x(k)\right) \delta x(k+1) \\ & = -\lambda^{\mathrm{T}}(t+N) E\left(x(t+N-1)\right) \delta x(t+N) \\ & - \sum_{k=t+1}^{t+N-1} \lambda^{\mathrm{T}}(k) E\left(x(k-1)\right) \delta x(k) \end{aligned} \tag{7}$$

Note that we set $\delta x(t) = 0$ in (7) because $\delta x(k)$ at $k = t$ is fixed as the current state. Taking the above relation into account, we obtain the variation in $\bar{J}$ as

$$\begin{aligned} & \delta\bar{J} = \\ & \sum_{k=t}^{t+N-1} \delta\lambda^{\mathrm{T}}(k+1) \left\{F\left(x(k), u(k)\right) - E\left(x(k)\right) x(k+1)\right\} \\ & + \left\{ \frac{\partial \phi}{\partial x(t+N)} - \lambda^{\mathrm{T}}(t+N) E\left(x(t+N-1)\right) \right\} \delta x(t+N) \\ & + \sum_{k=t+1}^{t+N-1} \left[\frac{\partial H}{\partial x(k)} - \lambda^{\mathrm{T}}(k+1) \frac{\partial\left\{E\left(x(k)\right) x(k+1)\right\}}{\partial x(k)} \right. \\ & \left. - \lambda^{\mathrm{T}}(k) E\left(x(k-1)\right) \right] \delta x(k) + \sum_{k=t}^{t+N-1} \frac{\partial H}{\partial u(k)} \delta u(k) \end{aligned} \tag{8}$$

On the basis of the variational principle, we obtain the necessary conditions for a stationary value of $\bar{J}$ over the horizon $(t \leq k \leq t+N)$ as follows.

$$E\left(x(k)\right)x(k+1) = F\left(x(k), u(k)\right) \tag{9a}$$

$$\lambda^{\mathrm{T}}(t+N)E\left(x(t+N-1)\right) = \frac{\partial \phi\left(x(t+N)\right)}{\partial x(t+N)} \tag{9b}$$

$$\lambda^{\mathrm{T}}(k)E\left(x(k-1)\right) = \frac{\partial H\left(x(k), \lambda(k+1), u(k)\right)}{\partial x(k)} - \lambda^{\mathrm{T}}(k+1)\frac{\partial\left\{E\left(x(k)\right)x(k+1)\right\}}{\partial x(k)} \tag{9c}$$

$$\frac{\partial H\left(x(k), \lambda(k+1), u(k)\right)}{\partial u(k)} = 0 \tag{9d}$$

Note that if $E\left(x(k)\right) = \mathrm{I}$, then the obtained stationary conditions (9) can be reduced to the well-known Euler–Lagrange equations [2]. Hence, we see that the stationary conditions (9) are natural extensions to the Euler–Lagrange equations. Accordingly, we call the stationary conditions (9) the generalized Euler–Lagrange equations.

Now, we can state the following theorem.

Theorem 1: The generalized Euler–Lagrange equations (9) must be satisfied for the performance index in (2) to be minimized subject to system equation (1).

Remark 1: Motivated by the fact that the cooling process of a hot strip mill [6] is governed by a nonlinear diffusion equation and its discretized equation with finite difference approximation belongs to a class of high-dimensional nonlinear implicit system (1), we consider the receding horizon control problem of system (1). In fact, it is meaningful to derive the stationary conditions (9) whichever $\det E\left(x(k)\right) = 0$ or not. When $\det E\left(x(k)\right) \neq 0$, implicit system (1) can be reduced to the explicit system as $x(t+1) = \hat{F}\left(x(t), u(t)\right)$, where $\hat{F}\left(x(t), u(t)\right) = E^{-1}\left(x(t)\right)F\left(x(t), u(t)\right)$. As the aforementioned example, in the case where $E\left(x(k)\right)$ has high dimensionality and complex nonlinearity, it is extremely difficult to calculate the Jacobian matrix of $\hat{F}\left(x(t), u(t)\right)$ for deriving the Euler–Lagrange equations, because $\hat{F}\left(x(t), u(t)\right)$ contains the inverse of $E\left(x(k)\right)$. It is important to note that the generalized Euler–Lagrange equations enable us to avoid this difficulty.

The remainder of this section is devoted to a brief description of the algorithm for numerically solving the generalized Euler–Lagrange equations (9).

Let $U(t) \in \mathbb{R}^{nN}$ be defined by

$$U(t) := [u^{\mathrm{T}}(t), u^{\mathrm{T}}(t+1), \cdots, u^{\mathrm{T}}(t+N-1)]^{\mathrm{T}}.$$

For a given initial optimal solution $U(t)$ and the present state $x(t)$, we first determine the state over the prediction horizon by using (9a), that is, $x(k)$ for $k = t, t+1, \cdots, t+N$ is calculated recursively from $k = t$ to $k = t+N$ by (9a). Next, the terminal costate $\lambda(t+N)$ is determined from the obtained terminal state $x(t+N)$ by (9b). Consequently, the costate over the prediction horizon is also determined by using (9c), that is, $\lambda(k)$ for $k = t+1, \cdots, t+N$ is calculated recursively from $k = t+N$ to $k = t+1$ by (9c). Figure 1 shows that the procedure for solving the equation of x is forward, whereas the one for solving the equation of λ is backward.

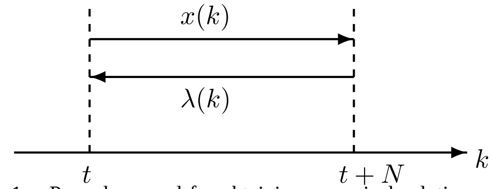

Fig. 1. Procedure used for obtaining numerical solutions.

Because $x(k)$ and $\lambda(k)$ for $k = t, t+1, \cdots, t+N$ are determined by $U(t)$ and $x(t)$ through (9a)-(9c), the remaining conditions (9d) for $k = t, t+1, \cdots, t+N$ can be regarded as a single equation,

$$D\left(U(t), x(t), t\right) := \begin{bmatrix} \frac{\partial H(x(t), \lambda(t+1), u(t))}{\partial u(t)} \\ \frac{\partial H(x(t+1), \lambda(t+2), u(t+1))}{\partial u(t+1)} \\ \vdots \\ \frac{\partial H(x(t+N-1), \lambda(t+N), u(t+N-1))}{\partial u(t+N-1)} \end{bmatrix} \tag{10}$$

Because $x(k)$ and $\lambda(k)$ are uniquely determined through (9a)-(9c) for given $U(t)$ and $x(t)$, $x(k)$ and $\lambda(k)$ depend on $U(t)$ and $x(t)$. Hence, it is reasonable to consider the arguments of D as $U(t)$, $x(t)$, t.

For given $U(t)$ and $x(t)$, D is not necessarily equal to zero, so $\|D\|$ is used to evaluate the optimality performance. If $\|D\| = 0$ is satisfied for the given $U(t)$ and $x(t)$, then the stationary conditions are satisfied. Several algorithms have been developed such that $\|D\|$ can be decreased by suitably updating $U(t)$, as discussed below.

A conventional way to update $U(t)$ is to replace $U(t)$ with $U(t) + \alpha s$, known as the steepest descent method, where s is the steepest descent direction and α is the step length satisfying the Armijo condition [9]. For Newton's method, s is given by the Hessian instead of the gradient. However, these methods are computationally expensive, and it was shown that the C/GMRES algorithm [2] is not only faster but also more numerically robust than these conventional algorithms. Next, a brief description of the C/GMRES method applied to this problem is provided. Instead of solving $D(U(t), x(t), t) = 0$ itself at each time by an iterative method such as the steepest descent method or Newton's method, we find the derivative of $U(t)$ with respect to time so that $D(U(t), x(t), t) = 0$ is satisfied identically. Namely we determine $\dot{U}(t)$ such that

$$\dot{D}\left(U(t), x(t), t\right) = -\xi D\left(U(t), x(t), t\right) \tag{11}$$

is satisfied, where ξ is a positive constant introduced to stabilize $D = 0$. If we choose $\xi = 1/\Delta t$, then the stability of (11) with forward difference approximation is guaranteed [2], where Δt denotes the sampling period. By total differentiation, we obtain

$$\frac{\partial D}{\partial U(t)}\dot{U}(t) = -\xi D - \frac{\partial D}{\partial x(t)}\dot{x}(t) - \frac{\partial D}{\partial t}. \tag{12}$$

This equation can be regarded as a linear algebraic equation with the coefficient matrix $(\partial D/\partial U(t))$, which can be used to determine $\dot{U}(t)$ for the given $U(t)$, $x(t)$, $\dot{x}(t)$, and t. Then, if the Jacobian $(\partial D/\partial U)$ is nonsingular, we obtain the following differential equation for $U(t)$:

$$\dot{U}(t) = \left(\frac{\partial D}{\partial U(t)}\right)^{-1} \left(-\xi D - \frac{\partial D}{\partial x(t)}\dot{x}(t) - \frac{\partial D}{\partial t}\right). \quad (13)$$

We can update the solution $U(t)$ of $D(U(t), x(t), t) = 0$ without using an iterative optimization method by integrating (13) in real time as, for example, $U(t+\Delta t) = U(t) + \dot{U}(t)\Delta t$. This approach is a type of continuation method [10] in the sense that the solution curve $U(t)$ is traced by integrating a differential equation. From the computational viewpoint, the differential equation (13) still involves expensive operations, i.e., solving the Jacobians $(\partial D/\partial U(t))$, $(\partial D/\partial x(t))$, and $(\partial D/\partial t)$ and linear algebraic equation associated with $(\partial D/\partial U)^{-1}$. To reduce the computational cost of the Jacobians and linear equation, we employ two techniques: the forward difference approximation for the products of the Jacobians and vectors and the GMRES method [11] for the linear algebraic equation. Using the forward difference approximation, we can obtain a linear equation with respect to $\dot{U}$. Thereafter, we can apply the GMRES algorithm to find the solution $\dot{U}(t)$ of the linear equation. Consequently, U can be updated so that $D = 0$ is stabilized. More detailed information about the implementation of C/GMRES is provided in [2].

Recently, we have developed a more efficient algorithm than C/GMRES, called the contraction mapping method [3]. In particular, we have proposed an algorithm for solving $D(U(t), x(t), t) = 0$ under the assumption that $D(U(t), x(t), t)$ satisfies a particular structural condition with respect to U. Compared with the C/GMRES method, the contraction mapping method has the disadvantage of limited applicability but the advantage of a smaller computational burden. More detailed information about the implementation of contraction mapping method is provided in [3].

IV. Stability of Closed-loop Systems

In this section, we consider the regularity (existence and uniqueness) of the solution of system equation (1) and the stability of the closed-loop system with receding horizon control. In the case where $E(x(t))$ is nonsingular for all $x(t)$, the existence and uniqueness of the solution can be guaranteed and the stability condition of the closed-loop system with receding horizon control has been established [1]. In this section, therefore, we consider the case of $\det E(x(t)) = 0$.

It is known that the regularity of the solution is effected by the rank of $E(x(t))$ [4]. Thus, it is difficult to analyze the solution of system (1) for the case where $\operatorname{rank} E(x(t))$ varies with $x(t)$. To avoid this difficulty, we impose the following assumption.

Assumption 1: There exist nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that

$$E(x(t)) = P \begin{bmatrix} \hat{E}(x(t)) & 0 \\ 0 & 0 \end{bmatrix} Q, \quad (14)$$

$$\det \hat{E}(x(t)) \neq 0, \quad (15)$$

$$P^{-1} F(x(t), u(t)) = \begin{bmatrix} G(x(t), u(t)) \\ R(x(t)) \end{bmatrix}, \quad (16)$$

are satisfied for all $x(t)$, where $\hat{E}(x(t)) : \mathbb{R}^n \to \mathbb{R}^{r\times r}$, $G(x(t), u(t)) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^r$, $R(x(t)) : \mathbb{R}^n \to \mathbb{R}^{(n-r)}$. Let $w \in \mathbb{R}^r$ and $z \in \mathbb{R}^{(n-r)}$ be defined by

$$\begin{bmatrix} w(t) \\ z(t) \end{bmatrix} := Qx(t). \quad (17)$$

Accordingly, let $\hat{E}(x(t))$, $G(x(t), u(t))$, and $R(x(t))$ be replaced with $\hat{E}(w(t), z(t)) : \mathbb{R}^r \times \mathbb{R}^{(n-r)} \to \mathbb{R}^{r\times r}$, $G(w(t), z(t), u(t)) : \mathbb{R}^r \times \mathbb{R}^{(n-r)} \times \mathbb{R}^m \to \mathbb{R}^r$, and $R(w(t), z(t)) : \mathbb{R}^r \times \mathbb{R}^{(n-r)} \to \mathbb{R}^{(n-r)}$, respectively.

Under Assumption 1, we can rewrite system (1) as the following form:

$$\begin{bmatrix} \hat{E}(w(t), z(t))\, w(t+1) \\ 0 \end{bmatrix} = \begin{bmatrix} G(w(t), z(t), u(t)) \\ R(w(t), z(t)) \end{bmatrix} \quad (18)$$

Remark 2: Assumption 1 implies that the system can be divided into the dynamical equation and the static constraint as shown in (18) and its structure doesn't vary with $x(t)$. Hence, there is a limitation on the variability of algebraic constraints imposed on the system. However, there many constrained systems that satisfy Assumption 1 in mechanical and electrical systems.

Next, the following assumption is imposed to guarantee the regularity of the solution.

Assumption 2: The Jacobian matrix $\partial R(w, z)/\partial z$ is nonsingular for all $w(t), z(t)$.

Here, we state the following theorem.

Theorem 2: Suppose that Assumptions 1-2 are satisfied. Then, there exists the unique solution of system (1).

Proof: Note that the regularity of solution of system (1) is equivalent to the one of system (18), because Q in (17) is assumed to be nonsingular. Consider x and y of $f(x, y)$ in Lemma 2 as $x = w$ and $y = z$, respectively. For given $w(t)$, it is obvious from Lemma 2 that there exists the unique solution $z(t) = g(w(t))$ satisfying $R(w(t), g(w(t))) = 0$. Consequently, $w(t+1)$ can be uniquely determined by $\hat{E}^{-1} G(w(t), g(w(t)), u(t))$. Likewise, for given $w(t+1)$, we can uniquely determine $z(t+1)$ using Lemma 2. Repeating this procedure, we can conclude that system (18) has a unique solution. This completes the proof. ■

Assumption 3: $G(0, 0, 0) = 0$ and $z = g(0) = 0$ at $x = 0, u = 0$, i.e., the origin of w and z is the equilibrium point.

Hereafter, we consider the stability of the closed-loop system with receding horizon control.

Let $X(t) \in \mathbb{R}^{n(N+1)}$ and $\Lambda(t) \in \mathbb{R}^{nN}$ be defined by

$$X(t) := \begin{bmatrix} x(t) \\ x(t+1) \\ \vdots \\ x(t+N) \end{bmatrix}, \quad \Lambda(t) := \begin{bmatrix} \lambda(t+1) \\ \lambda(t+1) \\ \vdots \\ \lambda(t+N) \end{bmatrix}. \quad (19)$$

Let $U^*(t)$ denote the sequence of the optimal control input over the prediction horizon defined by

$$U^*(t) := \begin{bmatrix} u^*(t) \\ \vdots \\ u^*(t+N-1) \end{bmatrix} := \arg\min_{U(t)} J\left(X(t), U(t)\right) \text{ subject to (1).} \quad (20)$$

Likewise, $X^*(t)$ and $\Lambda^*(t)$ denote the optimal state and costate sequence of the closed-loop system over the prediction horizon using $U^*(t)$, respectively.

Assumption 4: There exists the unique solution $X^*(t)$, $\Lambda^*(t)$, and $U^*(t)$ that satisfy generalized Euler–Lagrange equations (9).
Let a function $V\left(x(t)\right) : \mathbb{R}^n \to \mathbb{R}_+$ be defined by

$$V\left(x(t)\right) := J\left(X^*(t), U^*(t)\right). \quad (21)$$

Let $\hat{U}^*(t+1)$ be defined by

$$\hat{U}^*(t+1) := \begin{bmatrix} u^*(t+1) \\ \vdots \\ u^*(t+N-1) \\ u(t+N) \end{bmatrix}. \quad (22)$$

Therein, the final optimal control input $u^*(t+N)$ is replaced with any feasible control input $u(t+N)$. Accordingly, let $\hat{X}^*(t+1)$ be the state sequence of the closed-loop system using $\hat{U}^*(t+1)$.

Here, we introduce the well-known standard assumption for the stability analysis of the receding horizon control system.

Assumption 5: There exists a function $\alpha \in \mathbb{K}_\infty$ such that

$$V\left(x(t)\right) \leq \alpha\left(\|x(t)\|\right) \quad (23)$$

is satisfied for all $t \in \mathbb{Z}_+$.
Note that if there exists a positive constant ρ such that

$$\|u^*(t)\| \leq \rho \|x(t)\|$$

is satisfied for all $t \in \mathbb{Z}_+$, then Assumption 5 is satisfied. Thereby, Assumption 5 is called the weak controllability assumption [1].

Here, we provide the stability criteria for the closed-loop system using the receding horizon control.

Theorem 3: Under Assumptions 1–5, the closed-loop system using receding horizon control input $U^*(t)$ is asymptotically stable at the origin if there exists $u(t)$ such that the following inequality is satisfied for all $t \in \mathbb{Z}_+$.

$$\phi\left(x(t+1)\right) - \phi\left(x(t)\right) \leq -L\left(x(t), u(t)\right) \quad (24)$$

Proof: Because of Assumption 2, $z(t)$ and $x(t)$ can be given by

$$z(t) = g\left(w(t)\right), \quad x(t) = Q^{-1} \begin{bmatrix} w(t) \\ g\left(w(t)\right) \end{bmatrix} := h\left(w(t)\right)$$

Here, we consider the explicit system in (18).

$$w(t+1) = \hat{G}\left(w(t), u(t)\right), \quad (25)$$

where $\hat{G}\left(w(t), u(t)\right)$ is given by

$$\hat{G}\left(w(t), u(t)\right) := \hat{E}^{-1} G\left(w(t), g(w(t)), u(t)\right).$$

For system (25), we can apply Lemma 1 as shown below. Let $\hat{V}, \hat{\phi}, \hat{L}$ be given by $\hat{V}\left(w(t)\right) = V\left(h(w(t))\right)$, $\hat{\phi}\left(w(t)\right) = \phi\left(h(w(t))\right)$, $\hat{L}\left(w(t), u(t)\right) = L\left(h(w(t)), u(t)\right)$, respectively. Now, using the relation

$$J\left(X^*(t+1), U^*(t+1)\right) \leq J\left(\hat{X}^*(t+1), \hat{U}^*(t+1)\right) \quad (26)$$

we have the following:

$$\begin{aligned} \hat{V}\left(w(t+1)\right) &= \sum_{k=t+1}^{t+N} L\left(x^*(k), u^*(k)\right) + \phi\left(x^*(t+N+1)\right) \\ &\leq \sum_{k=t+1}^{t+N-1} L\left(x^*(k), u^*(k)\right) + L\left(x^*(t+N), u(t+N)\right) \\ &+ \phi\left(x(t+N+1)\right) =: \hat{V}^+\left(w(t+1)\right) \end{aligned} \quad (27)$$

Let $\hat{V}^+\left(w(t+1)\right)$ be defined as above. Using the above inequality, we have the following:

$$\begin{aligned} &\hat{V}\left(w(t+1)\right) - \hat{V}\left(w(t)\right) \leq \hat{V}^+\left(w(t+1)\right) - \hat{V}\left(w(t)\right) \\ &= -L\left(x(t), u^*(t)\right) + L\left(x^*(t+N), u(t+N)\right) \\ &+ \phi\left(x(t+N+1)\right) - \phi\left(x^*(t+N)\right) \end{aligned} \quad (28)$$

From the assumption in Theorem 3, we can see that there exists $u(t+N)$ such that the following inequality holds.

$$\begin{aligned} &\phi\left(x(t+N+1)\right) - \phi\left(x^*(t+N)\right) \\ &\leq -L\left(x^*(t+N), u(t+N)\right) \end{aligned} \quad (29)$$

Substituting (29) into (28) yields

$$\hat{V}\left(w(t+1)\right) - \hat{V}\left(w(t)\right) \leq -L\left(x(t), u^*(t)\right). \quad (30)$$

Here, note that there exists a positive constant ν such that the following inequalities hold.

$$\begin{aligned} &\hat{V}\left(w(t)\right) \geq L\left(x(t), u^*(t)\right) \\ &\geq L\left(h(w(t)), 0\right) \\ &\geq \nu \|w(t)\| \end{aligned} \quad (31)$$

Therefore, it follows that

$$\hat{V}\left(w(t+1)\right) - \hat{V}\left(w(t)\right) \leq -\nu \|w(t)\|. \quad (32)$$

Consequently, under Assumption 5, we can see that there exist $\mathbb{K}_\infty$ functions α_1 and α_2 such that the following inequalities are satisfied.

$$\alpha_1 (\|w(t)\|) \leq \hat{V} (w(t)) \leq \alpha_2 (\|w(t)\|)$$
$$\hat{V} (w(t+1)) - \hat{V} (w(t)) \leq -\alpha_1 (\|w(t)\|)$$

Hence, using Lemma 1, we can conclude that $w(t) = 0$ is asymptotically stable. Then, $z(t) = 0$ is also asymptotically stable because of Assumption 3. Consequently $x(t) = 0$ is asymptotically stable. This completes the proof. ■

Remark 3: The stabilization problem of nonlinear systems usually can be reduced to finding the so-called control Lyapunov function. Note that the difficulty of finding $u(t)$ satisfying inequality (24) in Theorem 3 is almost same as the difficulty of finding the control Lyapunov function ϕ.

V. Illustrative Example

In this section, we provide an illustrative example to verify the effectiveness of the proposed method. We consider RLC network circuits that can be described by the Brayton–Moser equations [7]. Let $n_L \in \mathbb{N}$ and $n_C \in \mathbb{N}$ denote the dimensions of circuit systems. Let $i_L \in \mathbb{R}^{n_L}$ and $v_C \in \mathbb{R}^{n_C}$ denote the inductor currents and capacitor voltages, respectively. Let the state $x \in \mathbb{R}^n$ be defined by $x := [i_L^{\mathrm{T}}, v_C^{\mathrm{T}}]^{\mathrm{T}}$, where $n = n_L + n_C$. Let $u \in \mathbb{R}^m$ denote the controlled voltage inputs. A discretized Brayton–Moser equation can be written as system model (1), where $E(x(t))$ and $F(x(t), u(t))$ are given by

$$E = \begin{bmatrix} A(i_L(t)) & 0 \\ 0 & C(v_C(t)) \end{bmatrix}$$

$$F = \begin{bmatrix} A(i_L(t))\, i_L(t) + \Delta t \left(\frac{\partial W(i_L, v_C)}{\partial i_L} \right)^{\mathrm{T}} \\ C(v_C(t))\, v_C(t) - \Delta t \left(\frac{\partial W(i_L, v_C)}{\partial v_C} \right)^{\mathrm{T}} + Bu \end{bmatrix}$$

Therein, $A(i_L(t)) \in \mathbb{R}^{n_L \times n_L}$ and $C(v_C(t)) \in \mathbb{R}^{n_C \times n_C}$ denote the inductance and capacitance matrices, respectively, both assumed to be positive definite. $W(i_L, v_C) : \mathbb{R}^{n_L} \times \mathbb{R}^{n_C} \to \mathbb{R}$ is a scalar function called the mixed potential function. $\Delta t \in \mathbb{R}_+$ and $B \in \mathbb{R}^{n_C \times m}$ denote the sampling time and input coefficient, respectively.

For sufficiently large n, it is impossible to compute the Jacobian matrix of $E^{-1}(x(t)) F(x(t), u(t))$, even if we utilize a symbolic math software such as Mathematica and Maple. In this case, the generalized Euler–Lagrange equations in (9) are useful to avoid this difficulty. However, we focus on the simplified model and provide easy simulation results. Here, we set system parameters as follow: $n_L = n_C = m = B = 1$,$\Delta t = 0.1$, $L(i_L) = 1 + i_L^2$, $C(v_C) = 1 + v_C^2$, $W(i_L, v_C) = i_L v_C$. Also, we consider performance index (2), where ϕ and L are given by $\phi := 0.5x_1^2 + 5x_2^2$ and $L := 0.5(x_1^2 + x_2^2 + u^2)$, respectively.

Figure 2 shows the free response of the states for the open-loop system without control inputs. In contrast, Fig. 3 shows the time response of the states for the closed-loop system with receding horizon control. This figure reveals that the states converge to zero and the system is asymptotically stable. The effectiveness of the proposed method was verified by numerical simulations.

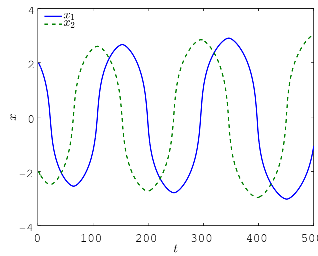

Fig. 2. Free response of $x(t)$ without control inputs.

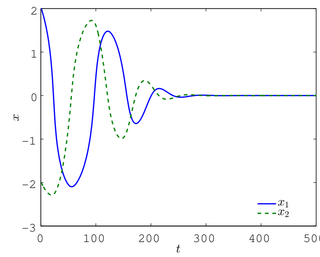

Fig. 3. Time response of $x(t)$ with receding horizon control.

VI. Conclusion

In this study, we provided a methodology to design a receding horizon controller for a generalized class of discrete-time nonlinear implicit systems. This method is advantageous for its applicability to a wide class of nonlinear implicit systems. For the optimal control problem, we derived the generalized Euler–Lagrange equations that must be satisfied for a performance index to be minimized. Moreover, we provided a brief description of numerical algorithms for solving the generalized Euler–Lagrange equations. For the stabilization problem, we established the stability criteria for the closed-loop nonlinear implicit systems with receding horizon control. Finally, we verified the effectiveness of the proposed method by numerical simulations.

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